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Continuous selections of solution sets of semilinear differential inclusions of fractional order

BY

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Abstract

In this thesis we will prove, in infinite dimensional spaces, an existence result concerning the existence of solutions for a semilinear differential inclusions with fractional order. Then we establish the existence of a continuous selection for the multifunction which represents the solution sets. We consider the case when there is a delay.

Our technique depends on using an appropriate fixed point theorem and a known result ensures the existence of continuous selections.

The obtained results extend a recent published result, concerning the existence of solutions for a semilinear differential inclusions with fractional order, from finite dimensional spaces to infinite dimensional spaces. Moreover, our technique allows to study the existence of continuous selections for some semilinear differential inclusions with fractional order in infinite dimensional spaces.

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0.1 General Introduction.

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, and so forth. For details, including some applications, see [27,36].

Elsayed And Ibrahim [21] initiated the existence of solutions for fractional differential inclusions. Many authors have been studied in recent papers the existence of solutions for differential inclusions or semilinear differential inclusions of fractional order, see for example [1,2,8,25,26,30,31,35,37,42,43].

Moreover, Breasan [11] discussed the existence of continuous selections of solution sets of differential inclusions of order one, Cernea [14] proved in finite dimensional, the existence of continuous selections of solutions sets of nonlinear integro-differential inclusions of fractional order $\alpha \in (1,2]$, Cernea [15] studied, in finite dimensional, the existence of continuous selections of solutions sets of fractional differential inclusions of fractional order $\alpha \in (1,2)$ involving Caputo's fractional derivative, Colombo [17] proved the existence of continuous selections of solution sets of Lipschitzean differential inclusions of order one, and Staicu [40] discussed the existence of continuous selections of solution sets to evolution equations. In addition, AL-Shary [23] proved the existence of continuous selections of solution sets to the following semilinear differential inclusion

$$(Q_\psi) \qquad \begin{cases} x'(t) \in Ax(t) + F(t, \tau(t)x), & a.e.\ t \in \ J, \\ x(t) = \psi(t), & t \in [-r, 0], \end{cases}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operator $\{T(t) : t \geq 0\}$ in E.

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Motivated by these works, in this thesis we have found, in infinite dimensional, the conditions that ensure the existence of continuous selections of solution sets for semilinear differential inclusion of fractional order and when the linear part is the infinitesimal generator of a semigroup of operators. To achieve this goal we extended, at first, a new result due to Ibrahim and Almoulhim [30] from finite dimensional to infinite dimension spaces. For more explanation, let $q \in (0,1]$, r, b be two positive real numbers, $J = [0,b]$, E be a real separable Banach space, $C_r = C([-r,0], E)$ be the Banach space of E-valued continuous functions on $[-r,0]$ with the uniform norm $\|x\| = \sup\{\|x(t)\|, \ t \in [-r,0]\}$, $C_b = C([-r,b], E)$, $F : J \times C_r \to P_{ck}(E)$ (the family of nonempty convex compact subsets of E) and $A : D(A) \subseteq E \to E$ be the infinitesimal generator of a C_0-semigroup of bounded linear operator $\{T(t) : t \geq 0\}$ in E. Consider the following fractional functional semilinear differential inclusion:

$$(P_\psi) \quad \begin{cases} D^q x(t) \in Ax(t) + F(t, \tau(t)x), & a.e.\ t \in J, \\ x(t) = \psi(t), & t \in [-r,0], \end{cases}$$

where ψ is a given element in $C_r = C([-r,0], E)$.

Let $S_\psi = \{x_\psi : x_\psi \text{ a solution for } (P_\psi)\}$. Note that $S_\psi \subseteq C_b$.

Our goals in this thesis are:

(1) Find the appropriate condition that ensure that the set S_ψ is nonempty.

(2) Find the appropriate condition that guarantee the existence of continuous selections for the multivalued function $\psi \to S_\psi$, that is, there is a continuous function $u : C_r \to C_b$ such that $u(\psi) \in S_\psi$.

The thesis is divided into two chapters. In the first chapter we present some fundamental concepts and facts related to set-valued functions, fractional calculus, semigroups and differential inclusions. In the second chapter we present our results.

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Introduction.

The aim of this chapter is to present some fundamental concepts and facts related to set-valued functions, fractional calculus and differential inclusions.

More precisely this chapter is organized as follows. In section 1, we will give some useful properties about Hausdorff metric topology. In section 2, we present the main notions and results concerning set-valued maps, their continuity, measurability and integrability. In section 3, we give the definitions and properties of fractional integral and fractional derivative in Riemann-Liouville and Caputo sense. In section 4, some facts about semigroups of linear operators are given. In sections 5 and 6, we present facts about differential inclusions and fractional functional differential inclusions. Finally, in section 7 we will give some important facts. Throughout this thesis we will use the following notations,

- $\mathcal{P}(E) = 2^E = \{A : A \subseteq E\}$,
- $\mathcal{P}_{cl}(E) = \{A \subseteq E : A$ is nonempty and closed of $E\}$,
- $\mathcal{P}_k(E) = \{A \subseteq E : A$ is nonempty and compact of $E\}$,
- $\mathcal{P}_{ck}(E) = \{A \subseteq E : A$ is nonempty, convex and compact of $E\}$,
- $\mathcal{P}_{cl,cv}(E) = \{A \subseteq E : A$ is nonempty, closed and convex of $E\}$,
- $\mathcal{P}_{bd}(E) = \{A \subseteq E : A$ is nonempty and bounded of $E\}$,
- $\mathcal{P}_{cb}(E) = \{A \subseteq E : A$ is nonempty, convex and bounded of $E\}$,
- $\mathcal{P}_{b\ cl\ c}(E) = \{A \subseteq E : A$ is nonempty, bounded, closed and convex of $E\}$.

1.1 Hausdorff Metric Topology.

The purpose of this section is to give material related to study the Hausdorff distance.

Definition 1.1.1 *(Definition 1.1.1 ,[13]). Let (X, d) be a metric space. If A , B $\in 2^X$, than we define*

(a) $e(A, B) = \sup\{d(a, B) : a \in A\}$ (the excess of A over B),

(b) $H(A, B) = \max\{e(A, B), e(B, A)\}$ (the Hausdorff distance between A and B).

Elementary properties:([13]). Let $A, B \in 2^X$:

1. $e(A,\phi)=\infty$ if $A \neq \phi$ and $e(\phi, B) = 0$ (by convention),
2. $e(A, B) = 0 \Leftrightarrow A \subseteq \overline{B}$,
3. $H(A, B) = 0 \Leftrightarrow \overline{A} = \overline{B}$,
4. $H(A, B) \leq H(A, C) + H(C, B)$, where $C \subseteq X$.

So by assertion 3, we see that the Hausdorff distance,

$$H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X) \to [0, \infty),$$

becomes a metric space. The topology induced by the Hausdorff distance is called the Hausdorff topology denotes by τ_H.

Definition 1.1.2 *(Definition 1.1.11,[28]). Let X be a normed space, X^* its (topological) dual and $A \in 2^X \setminus \{\phi\}$. The support function $\sigma(\cdot, A)$ of A is a function from X^* into $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ defined by $\sigma(x^*, A) = \sup\{\langle x^*, a\rangle : a \in A\}$, where $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair (X, X^*).*

Theorem 1.1.1 *(Th II2, Th II3, Th II5, ThII8, [13]).*

1. If $\{A_n, A\}_{n\in\mathbb{N}} \subseteq \mathcal{P}_{cl}(X)$ and $A_n \to A$ in the Hausdorff metric, then

$$A = \underset{n\geq 1}{\cap} \overline{\underset{m\geq n}{\cup} A_m},$$

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2. If (X,d) is a complete metric space, so is $(\mathcal{P}_{cl}(X),H)$,

3. If (X,d) is a complete metric space, then so is $(\mathcal{P}_{k}(X),H)$,

4. If X is a normed space, X^* its (topological) dual and $A,B\in\mathcal{P}_{b,\,cl,\,c}(X)$, then

$$H(A,B)=\sup\{|\delta^*(x^*,A)-\delta^*(x^*,B)|:\|x^*\|\leq 1\}.$$

5. If $(X,d\ )$ is a metric space and $A,B\in 2^X\setminus\{\phi\}$, then

$$H(A,B)=\sup\{|d\,(x,A)-d\,(x,B)|:x\in X\}.$$

Proof.

1. Let $B=\underset{n\geq 1}{\cap}\overline{\underset{m\geq n}{\cup}A_m}$. Let $\varepsilon>0$, $n\in\mathbb{N}$ and $x\in A$. There exists $m\geq n$ such that $H(A_m,A)\leq\varepsilon$, hence , $d(x,A_m)\leq\varepsilon$, and there exists a point $x_m\in A_m$ such that $d(x,x_m)\leq\varepsilon$. Therefore: $x\in\overline{\underset{m\geq n}{\cup}A_m}$, $\forall m,n\in\mathbb{N}$, that proves $A\subseteq B$. Let $x\in B$ and let us prove that $A_n\to A\cup\{x\}$ (that will proves $A\subseteq B$). From $A_n\to A$ follows: $e(A_n,A\cup\{x\})\to 0$, moreover we shall prove that $e(A\cup\{x\},A_n)=\max\{e(A,A_n),d(x,A_n)\}\to 0$. It is sufficient to prove $d(x,A_n)\to 0$. Let p such that $m,n\geq p\Rightarrow H(A_n,A_m)\leq\varepsilon$. From $x\in B$ follows that there exists $m\geq p$ such that: $d(x,A_m)\leq\varepsilon$. Hence if $n\geq p$, $d(x,A_n)\leq d(x,A_m)+H(A_m,A_n)\leq 2\varepsilon$.

2. Let $\{A_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{P}_{cl}(X),H)$. Let $A=\underset{n\geq 1}{\cap}\overline{\underset{m\geq n}{\cup}A_m}$. We will now show that $A\in\mathcal{P}_{cl}(X)$ and $A_n\to A$. First it is clear that A being the intersection of closed sets is closed, possibly empty. Let $\varepsilon>0$. Then for every $k\geq 0$, we can fined $N_k\geq 1$ such that $H(A_n,A_m)<\frac{\varepsilon}{2^{k+1}}$ for all $n,m\geq N_k$. Pick $n_0\geq N_0$ and $x_0\in A_{n_0}$. Then choose $n_1>\max[n_0,N_1]$ and $x_1\in A_{n_1}$ with $d(x_0,x_1)<\frac{\varepsilon}{2}$ (this is possible since $d(x_0,A_{n_1})\leq H(A_{n_0},A_{n_1})<\frac{\varepsilon}{2}$). Then if $\{n_k\}_{k\geq 0}$ is a strictly increasing sequence with $n_k\geq N_k$, inductively we can

generate a sequence $\{x_k\}_{k\geq 0} \subseteq X$ such that $x_k \in A_{n_k}$ and $d(x_k, x_{k+1}) < \frac{\varepsilon}{2^{k+1}}$. So $\{x_k\}_{k\geq 0}$ is a Cauchy sequence in X and since X is complete, we have that $x_k \to x \in X$. Because $\{n_k\}_{k\geq 0}$ is strictly increasing, given $n \geq 1$, we can find $k_n \geq 1$ such that $n_{k_n} \geq n$. Hence $x_k \in \underset{m\geq n}{\cup} A_m$ for $k \geq k_n$ and so $x \in \overline{\underset{m\geq n}{\cup} A_m}$ for all $n \geq 1$. Thus $x \in A$, which shows that $A \in \mathcal{P}_{cl}(X)$. In addition, we have

$$d(x, x_0) = \lim_{n\to\infty} d(x_n, x_0) \leq \lim_{n\to\infty} \sum_{k=1}^{n} d(x_k, x_{k-1}) < \varepsilon.$$

So for all $n_0 \geq N_0$ and all $x_0 \in A_{n_0}$, we have obtained an $x \in A$ such that $d(x, x_0) < \varepsilon$. Therefore $A_{n_0} \subseteq A_\varepsilon$. We need to show that $A \subseteq (A_n)_\varepsilon$ for all $n \geq N_0$. So let $x \in A$. Then $x \in \overline{\underset{m\geq N_0}{\cup} A_m}$ and we can find $m \geq N_0$ and $y \in A_m$ such that $d(x, y) < \frac{\varepsilon}{2}$. Also, if $n \geq N_0$, we have $d(x, A_n) \leq d(x, A_m) + H(A_m + A_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $H^*(A, A_n) < \varepsilon$ and this implies that $A \subseteq (A_n)_\varepsilon$ for $n \geq N_0$. Therefore finally we conclude that $A_n \to A$.

3. Let $\{A_n\}_{n\geq 1} \subseteq \mathcal{P}_k(X)$ and assume that $A_n \to A$. Then given $\varepsilon > 0$, we can find $n_0(\varepsilon) \geq 1$ such that for all $n \geq n_0(\varepsilon)$, $H(A_n, A) < \varepsilon$ and so $A \subseteq (A_n)_\varepsilon$. But by hypothesis A_n is compact, hence totally bounded. Thus we can find a finite set $F \subseteq X$ such that $A_n \subseteq F_\varepsilon$, hence $(A_n)_\varepsilon \subseteq F_{2\varepsilon}$. Therefore $A \subseteq F_{2\varepsilon}$ which shows that A is totally bounded and closed, hence $A \in \mathcal{P}_k(X)$.

4. The result is clearly true if $A = B$. So assume that $A \neq B$. Let $\varepsilon > 0$ such that $A \subseteq B + C_\varepsilon$ and $B \subseteq A + C_\epsilon$, where $C_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$ (hence $B + C_\varepsilon = B_\varepsilon$ and $A + C_\epsilon = A_\varepsilon$). Then for every $x^* \in X^*$ with $\|x^*\| \leq 1$, we have $\delta^*(x^*, A) \leq \delta^*(x^*, B) + \varepsilon$ and $\delta^*(x^*, B) \leq \delta^*(x^*, A) + \varepsilon$. Therefore $|\delta^*(x^*, A) - \delta^*(x^*, B)| \leq \varepsilon$ and so $\sup\{|\delta^*(x^*, A) - \delta^*(x^*, B)| : \|x^*\| \leq 1\} \leq H(A, B)$. On the other hand, if $\varepsilon = \sup_{\|x^*\|\leq 1} |\delta^*(x^*, A) - \delta^*(x^*, B)| > 0$, then we have $A \subseteq \overline{B + C_\varepsilon}$ and $B \subseteq \overline{A + C_\varepsilon}$. So $H(A, B) \leq \varepsilon$ and thus the formula follows.

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5. Note that for every $b \in B$, we have $d(x, A) \leq d(x, b) + d(b, A)$ and so $d(x, A) \leq d(x, B) + H(B, A)$. Similarly, we obtain that $d(x, B) \leq d(x, A) + H(B, A)$. Therefore

$$\sup\{|d(x, A) - d(x, B)| : x \in X\} \leq H(A, B).$$

On the other hand, we have

$$\begin{aligned} \sup\{d(b, A) \quad &: \quad b \in B\} = \sup\{d(b, A) - d(b, B) : b \in B\} \\ &\leq \quad \sup\{|d(x, A) - d(x, B)| : x \in X\} \end{aligned}$$

and

$$\sup\{d(a, B) : a \in A\} \leq \sup\{|d(x, A) - d(x, B)| : x \in X\}.$$

So it follows that

$$H(A, B) \leq \sup\{|d(x, A) - d(x, B)| : x \in X\}.$$

Then we conclude that

$$H(A, B) = \sup\{|d(x, A) - d(x, B)| : x \in X\}.$$

■

1.2 Set-Valued Analysis.

Let X and Y be two nonempty sets.

Definition 1.2.1 *([4]). A set-valued map F from X to Y is a map that associates with any $x \in X$ a subset $F(x)$ of Y. The subsets $F(x)$ are called the images or the values of F at x. The set-valued maps, are also called multifunctions, multivalued functions or point to set maps. We set*

$$\begin{aligned} Dom(F) &= \{x \in X : F(x) \neq \phi\}, \\ F(X) &= \underset{x \in X}{\cup} F(x) = \{y : \exists x \in X,\ y \in F(x)\}, \\ Gr(F) &= \{(x, y) \in X \times Y : y \in F(x)\}, \end{aligned}$$

the domain, range and graph of F. We say that F is homogeneous if $F(\lambda x) = \lambda F(x)$ *for any* $x \in X$ *and* $\lambda \in \mathbb{R}$.

The inverse F^{-} of the set-valued map F from X to Y is the set-valued map from the range of F to X defined by

$$x \in F^{-}(y) \Leftrightarrow y \in F(x) \Leftrightarrow (x, y) \in Gr(F).$$

The domain of F is thus the range of F^{-} and coincides with the projection of the graph of F into the space X, and in a symmetric way, the range of F is equal to the domain of F^{-} and to the projection of the graph of F into the space Y.

Definition 1.2.2 *([5], [32]). Let* X, Y *be topological spaces. A set-valued map* $F : X \to 2^{Y}$ *is said to be;*

i. Closed valued, open valued, compact valued and bounded valued if for all $x \in X$, $F(x)$ is a closed, open, compact and bounded set respectively in Y. Furthermore, if Y is a topological linear space and $\forall x \in X$, $F(x)$ is convex subset of Y, then $F(\cdot)$ is called convex valued.

ii. A closed, open set-valued map if and only if GrF is a closed (open) set in the product topology of $X \times Y$.

iii. Compact if $\underset{x \in X}{\cup} F(x)$ is relatively compact in Y.

iv. Locally compact if for any $x \in X$ have a neighborhood $V(x)$ such that $\underset{V(x)}{\cup} F(x)$ is relatively compact in Y.

Definition 1.2.3 *(Definition 2.4,[1]). Let* (X, d) *be a metric space. A set-valued map* $F : X \to 2^{X}$ *is said to be;*

i. Bounded on bounded sets if $F(B) = \underset{x\in B}{\cup} F(x) \in \mathcal{P}_{bd}(X)$ is bounded in X for all $B \in \mathcal{P}_{bd}(X)$, that is,

$$\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty.$$

ii. Completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$.

Definition 1.2.4 *(Page 36,[5]). Let $F : X \to 2^Y$ be a set-valued map. Then for any set $M \subseteq Y$, we have the following,*

- The lower inverse (weak inverse) image of M under $F(\cdot)$ is denoted by $F^-(M)$ and is defined as,

$$\begin{aligned} F^-(M) &= \{x \in X : F(x) \cap M \neq \phi\} \\ &= \underset{y\in M}{\cup} F^-(y). \end{aligned}$$

- The upper inverse (strong inverse or the core of the set M) image of M under $F(\cdot)$, denoted by $F^+(M)$ and is defined as $F^+(M) = \{x : F(x) \subseteq M\}$. Note that $F^-(M)$ and $F^+(M)$ coincide with the inverse of M viewing F as a function. If F has non-empty values then it is clear that $F^+(M) \subseteq F^-(M)$ for all subsets M of Y.

Example 1.2.1

1. The inverse of a non-bijective function. Let the function $f : \mathbb{R} \to \mathbb{R}^+$ be given by $f(x) = x^2$. For $y \in \mathbb{R}^+$,

$$f^{-1}(y) = \{-\sqrt{y}, \sqrt{y}\}.$$

2. $F : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, defined as:

$$F(x) = \begin{cases} [-1,1], & \text{if } x \text{ is integer,} \\ \{0\}, & \text{otherwise.} \end{cases}$$

is a set-valued map where $Dom\ (F) = \mathbb{R}$ and range of $F = [-1,1]$.

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Definition 1.2.5 *(Definition 8.2.1,[5]). Let (Ω, Σ) be a measurable space and X a complete separable metric space. Consider a set-valued map $F : \Omega \to 2^X$. A measurable map $f : \Omega \longrightarrow X$ satisfying*

$$\forall \omega \in \Omega, f(\omega) \in F(\omega),$$

is called a measurable selection.

Let $F : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})\backslash\{\phi\}$, be a set-valued map defined as $F(x) = [x, \infty)$, for all $x \in \mathbb{R}$. Then the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $f(x) = x$, for all $x \in \mathbb{R}$ is a selection of F.

Now we present the continuity concepts for set-valued maps which can be found in [29,34,38].

Definition 1.2.6 *(Definition 1.1.1,[4]). Let X and Y be two topological spaces and $F : X \longrightarrow P(Y)\backslash\{\phi\}$, be a set-valued map, we say that F is upper semicontinuous at $x_0 \in X$, u.s.c. for short, if for any open V containing $F(x_0)$ there exists a neighborhood $N(x_0)$ of x_0 such that $F(x) \subseteq V$ for all $x \in N(x_0)$. We say that F is upper semicontinuous if it is so at every $x_0 \in X$.*

It is known that (see [4]), the upper semicontinuity of a set-valued map $F : X \to 2^Y \setminus \{\phi\}$, is equivalent to any one of the two following conditions,

1. $F^+(A)$ is open in X whenever A is open in Y,
2. $F^-(A)$ is closed in X whenever A is closed in Y.

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Example 1.2.2

Let $F : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})\backslash\{\phi\}$, be a set-valued map defined as

$$F(t) = \begin{cases} [1,4] & \text{if} \quad t = 0, \\ [2,3] & \text{if} \quad t \neq 0. \end{cases}$$

To see the upper semicontinuity at $t = 0$, let $V \subseteq \mathbb{R}$ be a nonempty open set such that,

$$F(0) = [1,4] \subseteq V.$$

Take any neighborhood U of $t = 0$, then we have either

$$F(t) = [1,4] \text{ or } F(t) = [2,3]$$

for all $t \in U$. This implies, $\forall t \in U$:

$$F(t) \subseteq [1,4] \subseteq V.$$

Hence, F is $u.s.c.$ at $t = 0$.

In the following theorems, we collect some properties of $u.s.c.$ set-valued maps.

Theorem 1.2.1 *(Pro 2.17,[2]). Let X be a Hausdorff topological spaces, Y be a regular topological space and $F : X \to P(Y)\backslash\{\phi\}$. If F is upper semicontinuous with closed values, then the graph of F is closed in $X \times Y$.*

Proof. Let $(x_\alpha, y_\alpha)_{\alpha \in J} \subseteq GrF$ be a net converging to (x, y) in $X \times Y$. suppose $y \notin F(x)$. Since Y is regular, we can find two open sets $V_1, V_2 \subseteq Y$ such that $y \in V_1$, $F(x) \subseteq V_2$ and $V_1 \cap V_2 = \phi$. Because F is upper semicontinuous, we can find $\alpha_0 \in J$ such that for $\alpha \geq \alpha_0$, $F(\alpha_0) \subseteq V_2$, while $y_0 \in V_1$, a contradiction. ■

Example 1.2.3

The converse of the above theorem is not in general true. To see that, Let $X = Y = \mathbb{R}^+$ and define $F : X \longrightarrow \mathcal{P}_k(Y)$ by

$$F(x) = \begin{cases} [0, \frac{1}{x}] \text{ if } x > 0, \\ \{0\} \quad \text{if } \ x = 0. \end{cases}$$

Then clearly F is closed. However F is not upper semicontinuous at $x = 0$, since $F^+([0, 1)) = \{0\}$ is not open in X.

Theorem 1.2.2 *(Th 1.1,[2]). Let X and Y be two topological space such that $X \times Y$ is regular. Let F and G be two set-valued maps from X to Y such that $\forall x \in X$, $F(x) \cap G(x) \neq \phi$. suppose that:*

i) F is $u.s.c.$ at x_0.

ii) $F(x_0)$ is compact.

iii) The graph of G is closed.

Then, the set-valued map $F \cap G : x \rightarrow F(x) \cap G(x)$ is $u.s.c.$ at x_0.

Proof.

Let $N = N(F(x_0) \cap G(x_0))$ be an open neighborhood of $F(x_0) \cap G(x_0)$. We have to find a neighborhood $N(x_0)$ of x_0 such that $\forall x \in N(x_0)$, $F(x) \cap G(x) \subseteq N$. If $F(x_0) \subseteq N$, this follows from the upper semi-continuity of F. If $F(x_0) \nsubseteq N$, then we introduce the subset

$$K = F(x_0) \cap N^c$$

that is compact (since $F(x_0)$ is compact). Let $P = Graph(G)$, which is closed. For any $y \in K$, we have $y \notin G(x_0)$ and thus, $(x_0, y) \notin P$. Since P is closed and

$X \times Y$ is regular, there exists an open neighborhood $N_y(x_0)$ and $N(y)$ such that $P \cap (N_y(x_0) \times N(y)) = \phi$. Therefore

$$\forall x \in N_y(x_0), \quad G(x) \cap N(y) = \phi. \tag{1}$$

Since K is compact. it can be covered by n neighborhoods $N(y_i)$. The union $M = \overset{n}{\underset{i=1}{\cup}} N(y_i)$ is a neighborhood of K and $M \cup N$ is a neighborhood of $F(x_0)$. Since F is $u.s.c.$ at x_0, there exists a neighborhood $N_0(x_0)$ of x_0 such that

$$\forall x \in N_0(x_0), \quad F(x) \subseteq M \cup N. \tag{2}$$

We set $N(x_0) = N_0(x_0) \cap (\overset{n}{\underset{i=1}{\cap}} N_{y_i}(x_0))$. Hence, when $x \in N(x_0)$, then (1) and (2) imply that

$$\begin{cases} i)\ F(x) \subseteq M \cup N, \\ ii)\ G(x) \cap M = \phi. \end{cases}$$

Therefore, $F(x) \cap G(x) \subseteq N$ when $x \in N(x_0)$. ■

Corollary 1.2.1 *(Cor 1.1,[2]). Let X and Y be two topological spaces such that $X \times Y$ is regular. If Y is compact and the graph of G is closed in $X \times Y$, then G is upper semicontinuous.*

Proof.

We take F to be defined by $F(x) = Y$ for all $x \in X$ and we apply the above theorem. ■

Proposition 1.2.1 *(prop 1.3,[2]). Let F be an upper semi-continuous map with compact values from a compact space X to Y. Then $F(X) = \cup\{F(x) : x \in X\}$ is compact.*

Proof.

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We shall prove that any open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of $F(X)$ contains a finite subcover. Since each image $\{F(x)\}$ is compact, it can be covered by a finite number $n(x)$ of such U_λ. We write

$$F(x) \subseteq U_x = \underset{1\le i\le n(x)}{\cup} U_{\lambda_i}.$$

Since F is $u.s.c.$ at x, there exists an open neighborhood $N(x)$ of x such that $F(N(x)) \subseteq U_x$. But X is compact, it is contained in the union of p such neighborhoods $N(x_j)$. Thus

$$F(X) \subseteq \underset{1\le j\le p}{\cup} F(N(x_j)) \subseteq \underset{1\le j\le p}{\cup}\ \underset{1\le i\le n(x_j)}{\cup} U_{\lambda_i}.$$

Hence, from the open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ we have selected a finite open subcover $\{U_{\lambda_i} : 1 \le j \le p;\ 1 \le i \le n(x_i)\}$. This proves that $F(x)$ is compact. ■

Definition 1.2.7 *(Definition 1.1.2,[4]). Let X and Y be two topological spaces and $F : X \longrightarrow P(Y)\backslash\{\phi\}$. We say that F is lower semicontinuous at x_0, $l.s.c.$ for short, if for every open set V in Y with*

$$V \cap F(x_0) \neq \phi,$$

there exists a neighborhood $N(x_0)$ for x_0 such that

$$V \cap F(x) \neq \phi,$$

for all $x \in N(x_0)$. F is called lower semicontinuous if it is lower semicontinuous at each $x \in X$.

Remark 1.2.1 *(Page 43,[4]).*

The above definition could be phrased by means of generalized sequences as follows: given any generalized sequence x_μ converging to x_0 and any $y_0 \in F(x_0)$, then there exists a generalized sequence $y_\mu \in F(x_\mu)$ that converges to y_0. When

X and Y are metric spaces, this last characterization holds true with usual (i.e. countable) sequence.

Evidently, the lower semicontinuity of a set-valued map $F : X \longrightarrow P(Y)\backslash\{\phi\}$, where X and Y are two topological spaces, is equivalent to any one of the following conditions:

1. $F^{-}(A)$ is open in X whenever A is open in Y ,
2. $F^{+}(A)$ is closed in X whenever A is closed in Y.

Example 1.2.4

1. Let $F : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})\backslash\{\phi\}$, be a set-valued map defined as
$$F(t) = \begin{cases} [1,4] & \text{if} \quad t = 0, \\ [2,3] & \text{if} \quad t \neq 0. \end{cases}$$
It is not $l.s.c.$ at $t = 0$. To see that if $r \in F(0) = [1,4]$ such that $3 < r < 4$ and
$$V = (r - \epsilon, r + \epsilon) \subseteq (3,4),$$
for a sufficiently small $\epsilon > 0$, there exists a neighborhood U of $t = 0$ such that for some $t \in U(0)$ we have $F(t) \cap V = \phi$. Note that F is upper semicontinuous at $t = 0$.

2. Let $F : \mathbb{R} \longrightarrow P(\mathbb{R})\backslash\{\phi\}$, be a set-valued map defined as
$$F(t) = \begin{cases} [0,1] & \text{if} \quad t \neq 0, \\ \{\frac{1}{2}\} & \text{if} \quad t = 0. \end{cases}$$

Clearly F is lower semicontinuous. Notice that F does not have closed graph. So, we need not think about characterization in terms of GrF which is indicated in $u.s.c.$ case. So we have the following important characterization of lower semicontinuous.

Proposition 1.2.2 *(Proposition 1.2.26,[28]). Let X be Hausdorff topological space and Y be a metric space and $F : X \longrightarrow P(Y)\backslash\{\phi\}$, then F is l.s.c., iff for every $y \in Y$,*

$$x \rightarrow \varphi_y = d(y, F(x))$$

is u.s.c.

Proof.

We will show that for every $\lambda \in \mathbb{R}$, the upper level set $U_\lambda = \{x \in X : \varphi_v(x) \geq \lambda\}$ is closed. Indeed let $\{x_n\}_{n\in I} \subseteq U_\lambda$ be a net and assume that $x_n \rightarrow x$ in X. Given $\varepsilon > 0$ we can find $y \in F(x)$ such that $d(v, y) \leq \varphi_v(x) + \varepsilon$. Also since $F(.)$ is $l.s.c.$, we can find $n_0 \in I$ such that for all $n \geq n_0$ we have $F(x_n) \cap B(y, \varepsilon) \neq \phi$. Thus we can find $y_n \in F(x_n)$ such that $d(v, y_n) \leq \varphi_v(x) + 2\varepsilon$ and so $\varphi_v(x_n) \leq \varphi_v(x) + 2\varepsilon$, which in turn implies that $\lambda \leq \varphi_v(x) + 2\varepsilon$. Let $\varepsilon \downarrow 0$, to obtain $\lambda \leq \varphi_v(x)$. Thus $x \in U_\lambda$ and so $\varphi_v(.)$ is $u.s.c.$

We need to show that if $V \subseteq Y$ is open, $F^-(V)$ is open in X. Let $x \in F^-(V)$. Then $F(x) \cap V \neq \phi$ and choose $y \in F(x) \cap V$. Then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq V$. Also since $\varphi_y(.)$ is $u.s.c.$, given this $\varepsilon > 0$, we can find $U \in N(x)$ such that $\varphi_y(x') < \varphi_y(x) + \varepsilon = \varepsilon$ for all $x' \in U$. Hence $F(x') \cap B(y, \varepsilon) \neq \phi$ for all $x' \in U$ and so $F(x') \cap V \neq \phi$ for all $x' \in U$. Therefore $U \subseteq F^-(V)$ which means that $F^-(V)$ is open and so $F(.)$ is $l.s.c.$ ■

Definition 1.2.8 *(Definition 1.1.3,[4]). Let X and Y be two topological spaces. A set-valued map $F : X \longrightarrow P(Y)\backslash\{\phi\}$ is said to be continuous at $x_0 \in X$ if it is both u.s.c., and l.s.c. at x_0. It is said to be continuous if it is continuous at every point $x \in X$.*

Definition 1.2.9 *(Definition 1.1.2,[19]). Let X and Y be two normed spaces. We say that F is $\varepsilon-\delta-$u.s.c. at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|x - x_0\| < \delta \Rightarrow F(x) \subseteq B_Y(F(x_0), \varepsilon),$$

which is equivalent to

$$\|x - x_0\| < \delta \Rightarrow e((F(x), F(x_0) < \varepsilon.$$

We say that F is $\varepsilon-\delta-l.s.c.$ at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow F(x_0) \subseteq B_Y(F(x), \varepsilon),$$

which is equivalent to

$$\|x - x_0\| < \delta \Rightarrow e((F(x_0), F(x) < \varepsilon.$$

Proposition 1.2.3 *(Proposition.1.1, Proposition.2.1,[24]). Let X and Y be two normed spaces, F is a set-valued map from X to Y. Then we have:*

1. If F is u.s.c., then F is $\varepsilon - \delta - u.s.c.$ The converse is true if F has compact values.

2. If F is $\varepsilon - \delta - l.s.c.$, then F is $l.s.c.$ The converse is true if F has compact values.

Remark 1.2.2 *(Example1.2.62,[28],Page 45,[4]).*

1. To show that the converse of the assertion (1) in the preceding proposition is not true if the values of F are not compact consider the following Example. Let $X = [0,1]$, $Y = \mathbb{R}$ and let $F : X \longrightarrow 2^Y \setminus \{\phi\}$ be defined by

$$F(x) = \begin{cases} [0,1] & \text{if } 0 \leq x < 1, \\ [0,1) & \text{if } \qquad x = 1. \end{cases}$$

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It easy to check that F is $\varepsilon-\delta-u.s.c.$ but not $u.s.c.$ at $x=1$. Indeed note that $F^{+}[(-1,1)]=\{1\}$ not an open set.

2. To show that the converse of the assertion (2) in the preceding proposition is not true if the values of F are not compact consider the following Example. Let $F:\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R}^2)\setminus\{\phi\}$ be defined by

$$F(t)=\{(t,xt):t\in\mathbb{R}\}.$$

Obviously $F(\cdot)$, is $l.s.c.$ but not $\varepsilon-\delta-l.s.c.$, since for any $t\neq t_0, v=(x,y)$

$$\sup\{d(v,F(t)):v\in F(t_0)\}=\infty,$$

which implies that there is $\epsilon>0$ such that

$$e\left(F(t_0),F(t)\right)\geq\epsilon.$$

Definition 1.2.10 *(2.6,[1]). Let E be a normed space. A multivalued function $F:E\to\mathcal{P}_{cl}(E)$ is called,*

(i) γ-Lipschitz if and only if there exists $\gamma>0$, such that

$$H(F(x),F(y))\leq\gamma d(x,y).$$

(ii) Contraction if and only if it is $\gamma-$Lipschitz with $\gamma<1$.

(iii) F has fixed point if there exists $x\in E$, such that $x\in F(x)$.

1.2.1 Measurable multifunctions.

Definition 1.2.1.1 **(**Definition 2.1.1,[28]**).** Let (Ω,Σ) be a measurable space and (X,d) a separable metric space and $F:\Omega\to 2^X\setminus\{\phi\}$ be a multifunction. F is said to be:

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1. "Strongly measurable" if for every $U \subseteq X$ closed, we have

$$F^{-}(U) = \{w \in \Omega : F(w) \cap U \neq \phi\} \in \Sigma,$$

2. " Measurable" if for every $U \subseteq X$ open, we have

$$F^{-}(U) = \{w \in \Omega : F(w) \cap U \neq \phi\} \in \Sigma,$$

3. "Graph measurable" if

$$GrF = \{(w,x) \in \Omega \times X : x \in F(w)\} \in \Sigma \times \mathcal{B}(X),$$

where $\Sigma \otimes \mathcal{B}(X)$ is a product σ- algebra on $\Omega \times X$ (i.e. the smallest σ- algebra containing all products $A \times B$, with $A \in \Sigma$, $B \in \mathcal{B}(X)$).

Remark 1.2.1.1 (Proposition 2.1.8,[28]).

Recalling that for $U \subseteq X$ open we have $F(w) \cap U \neq \phi$ if and only if $\overline{F(w)} \cap U \neq \phi$. Thus $F : X \to 2^{Y}$ is measurable if and only if $\overline{F} : X \to \widehat{\mathcal{P}}_{cl}(X)$ is measurable.

Proposition 1.2.1.1 (Proposition 2.1.13, Proposition 2.1.14, [28]). Let (Ω, Σ) be a measurable space and (X, d) a separable metric space, $F : \Omega \to \mathcal{P}_{cl}(X)$,

1. Strongly measurability$\Rightarrow$ measurability.

2. If

$$X = \bigcup_{n=1}^{\infty} K_n,$$

where K_n are compact sets (i.e., X is σ-compact) then the data in assertion 1 are equivalent.

Proof.

1. Let $U \subseteq X$ be open. Recall that in a metrizable space, every open set is an F_σ-set (i.e., countable union of closed sets). So $U = \bigcup_{n \geq 1} C_n$, $C_n \subseteq X$

closed $n \geq 1$. We have $F^{-}(U) = F^{-}(\underset{n\geq 1}{\cup} C_n) = \underset{n\geq 1}{\cup} F^{-}(C_n) \in \sum$. Hence $F(.)$ is measurable.

2.Let $C \subseteq X$ be closed and let $X = \overset{\infty}{\underset{n=1}{\cup}} K_n$ where $K_n \subseteq X$ compact. Then $F^{-}(C) = F^{-}(\overset{\infty}{\underset{n=1}{\cup}} C \cap K_n) = \overset{\infty}{\underset{n=1}{\cup}} F^{-}(C \cap K_n) \in \sum$. That means that F is strongly measurable, then measurable. ■

Proposition 1.2.1.2 (Proposition 2.1.4**,** Proposition 2.1.7, [28]). Let (Ω, Σ) be a measurable space and (X, d) a separable metric space, $F : \Omega \to 2^X$ then,

1. F measurable if and only if

$$\forall x \in X, w \to d\left(x, F\left(w\right)\right) = \inf\left[d\left(x, x'\right) : x' \in F\left(w\right)\right],$$

is a measurable $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$-valued function.

2. If $F(\cdot) \in \widehat{\mathcal{P}}_{cl}(X)$ is measurable, then GrF is measurable.

Proof.

1. Let $\lambda > 0$ and define $L_\lambda(x) = \{w \in \Omega : d(x, F(w)) < \lambda\}$. Then if $U = B(x, \lambda)$, we have $L_\lambda(x) = F^{-}(B(x, \lambda)) \in \sum$ and so we conclude that $w \to d(x, F(w))$ is measurable.

Now for every $x \in X$ and every $\lambda > 0$, we have $F^{-}(B(x, \lambda)) = L_\lambda(x) \in \sum$. Now let $U \subseteq X$ be open. Then because X is separable $U = \underset{n\geq 1}{\cup} B(x_n, \lambda_n)$ and so $F^{-}(U) = \underset{n\geq 1}{\cup} F^{-}(B(x_n, \lambda_n)) \in \sum$. Therefore $F(.)$ is measurable.

2. Since $F(.)$ is closed valued we have that $GrF = \{(w, x) \in \Omega \times X : d(x, F(w)) = 0\}$. But $(w, x) \to d(x, F(w))$ is measurable. Therefore $GrF \in \sum \times \mathcal{B}(X)$. ■

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1.2.2 Measurable selections. Measurable multifunctions with values in complete separable metric space.

Theorem 1.2.2.1 (Selection Theorem) (Th 8.1.3., [5])**.** Let (Ω, Σ) be a measurable space, X be a complete metric space and $F : \Omega \to \mathcal{P}_{cl}(X)$ is measurable, then F admits a measurable selection (i.e., there exists $f : \Omega \to X$ measurable such that for every $\omega \in \Omega$, $f(\omega) \in F(\omega)$).

Proof.

Let $d(\cdot,\cdot)$ be a complete metric on X defining its topology. Without any loss of generality, we may assume that the $d$diameter of X is strictly less than one. Let $\{x_n\}_{n\geq 1}$ be a countable dense subset of X. $\forall\omega \in \Omega$, let $n \geq 1$ be the smallest integer such that $F(\omega) \cap B^{\circ}(x_n, 1) \neq \phi$. We set $f_0(\omega) = x_n$. Then $f_0(\cdot)$ is measurable. Furthermore,

$$\forall\omega \in \Omega, d(f_0(\omega), F(\omega)) < 1.$$

Assume that we already constructed measurable maps .We will construct a sequence of measurable maps $f_n : \Omega \to X$ such that

$$\begin{aligned} f_k &: \Omega \to X, \\ f_k &= x_n, k = 0, \cdots, m, \end{aligned}$$

satisfying

(i) $d(f_k(\omega), F(\omega)) < \dfrac{1}{2^k}$ for every $0 \leq k \leq m$ and every $\omega \in \Omega$ and

(ii) $d(f_k(\omega), f_{k+1}(\omega)) < \dfrac{1}{2^{k-1}}$ for every $0 \leq k < m-1$, and every $\omega \in \Omega$. $\forall n \geq 1$

$$D_n = \{\omega \in \Omega : f_m(\omega) = x_n\}.$$

The sets D_n is disjoint and $\Omega = \underset{n\geq 1}{\cup} D_n$. Furthermore (i) implies that

$$\forall\omega \in \Omega, F(\omega) \cap B^{\circ}(x_n, 2^{-m}) \neq \phi.$$

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Fix $\omega \in \Omega$ and let n be such that $\omega \in D_n$. Consider the smallest integer r such that

$$F(\omega) \cap B^{\circ}(x_n, 2^{-m}) \cap B^{\circ}(x_r, 2^{-(m+1)}) \neq \phi,$$

and set $f_{m+1} = x_r$. Then by the induction hypothesis, we can find $z \in F(\omega)$ such that $d(f_{m+1}(\omega), z) < 2^{-m+1}$. Since $\{x_n\}_{n\geq 1}$ is dense in X , there is $n \geq 1$ such that $d(x_n, z) < 2^{-m}$ and

$$\begin{aligned} d(f_m(\omega), f_{m+1}(\omega)) &\leq 2^{-m} + 2^{-(m+1)} \\ &< 2^{-m+1}. \end{aligned}$$

Clearly, this defines a measurable map

$$f_{m+1} : \Omega \to X, f_{m+1} = x_n$$

and (i),(ii) hold true with m replaced by $m+1$. Moreover from property (ii), we see that $\{f_k(\omega)\}_{k\geq 0}$ is Cauchy in X (in fact, uniformly in $\omega \in \Omega$). Since X is complete, there exists $f : \Omega \to X$ such that $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$ and of course, $f(\cdot)$ is measurable. Finally from property (i), we have that $d(f(\omega), F(\omega)) = 0$ for all $\omega \in \Omega$ and so $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. ■

Theorem 1.2.2.2 (Th 2.2.4, [28]**).** Let (Ω, Σ) be a measurable space with $\mu \geq 0$, X a separable metric space and $F : \Omega \to \mathcal{P}_{cl}(X)$. Consider the following statements:

1. $F^{-}(D) \in \Sigma$ for all $D \in \mathcal{B}(X)$,
2. F is strongly measurable,
3. F is measurable,
4. For all $x \in X$, $\omega \to d(x, F(\omega))$ is measurable,

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5. F has Casting representation, that is: there exist a sequence $\{f_n\}_{n\geq 1}$ of measurable selections of F such that

$$\forall \omega \in \Omega, F(\omega) = \overline{\underset{n\geq 1}{\cup}\{f_n(\omega)\}}.$$

6. $GrF \in \Sigma \times \mathcal{B}(X)$.

Then the following are true:

(a) (1)⇒(2)⇒(3)⇔(4)⇒(6).

(b) If X is complete, then (3)⇔(5).

(c) If X is σ-compact, then (2)⇔(3).

(d) If (Ω, Σ, μ) is a complete $\sigma-$ finite measure space and X is a complete space, then (1)⇒(6) are all equivalent.

1.2.3 Calculus of Measurable Multifunctions.

In this part we will present the most important theories that help us to calculate of measurable of multifunctions.

Theorem 1.2.3.1 **"**Union and Intersection" (Propstion.III.4,[16], Th 8.2.4, [5]). Let (Ω, Σ) be a measurable space and X be a metrizable separable space, $F_n : \Omega \to \mathcal{P}_k(X), n \in \mathbb{N}$. Then if F_1, F_2 are measurable multifunctions the multifunction $\omega \to F_1(\omega) \cap F_2(\omega)$ is measurable, more generally $t \to \cap F_n(t)$ is measurable and if $\overline{\cup F_n(\omega)}$ is compact, then $\omega \to \overline{\cup F_n(\omega)}$ is measurable. Moreover if (Ω, Σ) be a complete σ-finite measure space and X be a Polish space then $\cap F_n(\omega)$ and $\overline{\cup F_n(\omega)}$, are measurable.

Lemma 1.2.3.1 "Direct Image" (Theorem 8.2.8, [5]). Let (Ω, A, μ) be a complete $\sigma-$finite measure space, X a complete separable metric space and $F : \Omega \to$

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2^X with nonempty closed images. Consider a multivalued function G from: $\Omega \times X$ to $P(Y)$,Y is a *complete separable metric* space such that for every $x \in X$ the multivalued function $w \to G(w,x)$ is measurable and for every $w \in \Omega$ the multivalued function $x \to G(w,x)$ is continuous. Then the multivalued function $w \to \overline{G(w,F(w))}$ is measurable. In particular for every measurable single-valued function $z : \Omega \to X$, the multivalued function $w \to G(w, z(w))$ is measurable and for every Carathéodory single-valued function $\varphi : \Omega \times X \to Y$, the multivalued function $w \to \overline{\varphi(w, F(w))}$ is measurable.

Proof.

It is not restrictive to assume that F has nonempty images. Then, there exists a dense sequence $(f_n)_{n\geq 1}$ of measurable selections of F.

We claim that the map $w \to G(w, f_n(w))$ is measurable. Indeed consider a sequence of measurable simple maps f_{nk} from Ω to X, converging pointwise to f_n when $k \to \infty$. Then, since f_{nk} are simple, for every k the set-valued map $w \to G(w, f_{nk}(w))$ is measurable. On the other hand, since $G(w, .)$ is continuous,

$$\forall w \in \Omega, \lim_{k\to\infty} G(w, f_{nk}(w)) = G(w, f_n(w))$$

and we deduce that this limit is again a measurable map.

On the other hand, since G is continuous with respect to the second variable and $(f_n)_{n\geq 1}$ is dense, for every $w \in \Omega$

$$G(w, F(w)) = \overline{\bigcup_{n\geq 1} G(w, f_n(w))}$$

Theorem (1.2.3.1) ends the proof. ■

Theorem 1.2.3.2 **(ThIII.41, [13]).** Let (Ω, Σ) be a complete $\sigma-$finite measure space, and E a separable Banach space. Let $f : \Omega \to E$ be a measurable map and $\rho : \Omega \to [0, \infty)$ a measurable function. Then

1. $\omega \to B(f(\omega), \rho(\omega))$ (the closed ball with center$f(\omega)$ and radius $\rho(\omega)$ is a measurable multifunction),

2. If $F : \Omega \to \mathcal{P}_{cl}(E)$ is measurable such that for all $\omega \in \Omega$, the set

$$\Sigma(\omega) = \{x \in F(\omega) : \|f(\omega) - x\| = d(f(\omega), F(\omega))\},$$

   is nonempty, then there is a measurable function $g : \Omega \to E$ such that $g(\omega) \in \Sigma(\omega)$, a.e. , and

$$\|f(\omega) - g(\omega)\| = d(f(\omega), F(\omega)), \forall \omega \in \Omega.$$

1.2.4 Integration of set-valued maps.

This section deals with the integration of measurable multifunctions. Throughout this section, let (Ω, Σ, μ) be a complete $\sigma-$finite measure space and X a separable Banach space supplied with the norm $\|\cdot\|$. A single-valued function $f : \Omega \longrightarrow X$ is called "Bochner" integrable if f is strongly measurable (the $a.e.$-limit of step functions f_n) and the function $\omega \longrightarrow \|f(\omega)\|$ is Lebesgue integrable. The integral of f is defined as

$$\int_E f(\omega) d\mu(\omega) = \lim_{n\to\infty} \int_E f_n(\omega) d\mu(\omega), \forall E \in \Sigma.$$

For every p, $1 \leq p < \infty$ let $L^p(\Omega, \Sigma, \mu)$ be the vector space of all equivalence classes with the norm

$$\|f\|_p = (\int_\Omega \|f\|^p \, d\mu)^{1/p}.$$

It is known that $L^p(\Omega, \Sigma, \mu)$; for some $p \geq 1$ is a Banach space.

Let F be a set-valued map from Ω to the nonempty closed subsets of X. F is called "integrably bounded" if there exists a nonnegative function $k \in L^1(\Omega, \Sigma, \mu)$ such that

$$\forall f(\omega) \in F(\omega), \|f(\omega)\| \leq k(\omega) \ a.e. \text{ in } \Omega.$$

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Clearly if F is measurable and integrably bounded, then the family

$$S_F^1 = \{f \in L^1(\Omega, \Sigma, \mu) : f(\omega) \in F(\omega) \ a.e. \text{ in } \Omega\}$$

is not empty. Let $F : \Omega \to 2^X \setminus \{\phi\}$, be a set valued map with $S_F^1 \neq \phi$. Then the set valued map (Aumman) integral of F is defined in the following (see [16]).

Definition 1.2.4.1 Let $F : \Omega \to \mathcal{P}_{cl}(X)$ be a measurable set-valued map and $S_F^1 \neq \phi$. Then the set-valued (Aumman) integral of F is defined in the following

$$\int_\Omega F(\omega) d\mu(\omega) = \{\int_\Omega f(\omega) d\mu(\omega) : f \in S_F^1\}.$$

In the following theorem, we collect some properties for the integral of F.

Theorem 1.2.4.1 (Propstion.8.6.2, Theorem8.6.3, Theorem8.6.4, [5]). If $F, F_j : \Omega \to \mathcal{P}_{cl}(X), j = 1, 2$ are measurable multivalued maps with $S_F^1, S_{Fj}^1 \neq \phi$, and set $G(\omega) = \overline{F_1(\omega) + F_2(\omega)}$. Then

1. If F has convex images then the integral $\int_\Omega F(\omega) d\mu(\omega)$ is convex.
2. $\int_\Omega \lambda F(\omega) d\mu(\omega) = \lambda \int_\Omega F(\omega) d\mu(\omega)$, for all $\lambda \in \mathbb{R}$.
3. $\overline{\int_\Omega \overline{co} F(\omega) d\mu}(\omega) = \overline{co} \int_\Omega F(\omega) d\mu(\omega)$.
4. $\overline{\int_\Omega G(\omega) d\mu}(\omega) = \overline{\int_\Omega F_1(\omega) d\mu(\omega) + \int_\Omega F_2(\omega) d\mu(\omega)}$.
5. If the values of F are convex compact subsets of X, then $\int_\Omega F(\omega) d\mu(\omega)$ is convex compact subset of X.
6. If $X = \mathbb{R}^n$ and μ is nonatomic (i.e. Σ does not contain atoms, where $A \in \Sigma$ is called an "atom" if $\mu(A) > 0$ and for every measurable subset $A_1 \subseteq A$, $\mu(A_1)$ is equal to either 0 or $\mu(A)$), then $\int_\Omega F(\omega) d\mu(\omega)$ is compact convex and the extremal points of $\overline{co}(\int F(\omega) d\mu(\omega))$ are contained in $\int_\Omega F(\omega) d\mu(\omega)$.

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7. If μ is nonatomic, then

$$\overline{\int_\Omega F(\omega)\, d\mu(\omega)} = \overline{co}\int_\Omega F(\omega)\, d\mu(\omega) = \int_\Omega \overline{co}\ F(\omega)\, d\mu(\omega).$$

Furthermore, when X is reflexive and F has convex images, then

$$\int_\Omega F(\omega)\, d\mu(\omega),$$

is closed.

Proof. 3. It is clear that

$$\overline{co}\int_\Omega F d\mu \subset \overline{\int_\Omega \overline{co}\ F d\mu}.$$

To prove the converse inclusion, fix $k \in L^1(\Omega, \mathbb{R}, \mu)$ with strictly positive values and a measurable selection g of $\overline{co}F$. The support function

$$\omega \to \sigma(\overline{co}\ F(\omega), p) = \sigma(F(\omega), p)$$

is measurable and observe that for every $p \in X^*$

$$(p, \int_\Omega g d\mu) \leq \int_\Omega \sigma(\overline{co}\ F(\omega), p)\mu(d\omega) = \int_\Omega \sigma(F(\omega), p)\mu(d\omega).$$

Fix $\varepsilon > 0$ and set $\phi(x) = (p, x)$. On the measurability of inverse images applied with

$$G(\omega) = [\sigma(F(\omega), p) - \varepsilon k(\omega), \sigma(F(\omega), p)]$$

yields that there exists a measurable selection $f \in F$ such that

$$(p, f(\omega)) \geq \sigma(F(\omega), p) - \varepsilon k(\omega).$$

Consequently

$$\int_\Omega \sigma(F(\omega), p)\mu(d\omega) \leq \int_\Omega ((p, f) + \varepsilon k) d\mu \leq \sigma(\int F d\mu, p) + \varepsilon \|k\|_{L^1}.$$

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Since $\varepsilon > 0$ and $p \in X^*$ are arbitrary, using the separation theorem, we end the proof.

4. To prove the equality it is enough to show that

$$\int_\Omega G d\mu \subset \overline{\int_\Omega F_1(\omega)\, d\mu(\omega) + \int_\Omega F_2(\omega)\, d\mu(\omega)}$$

since the other inclusion is obvious.

Consider a measurable selection g of G and let $(f_{in})_{n\geq 1}$ be dense sequence of measurable selections of F_i. Then $(f_{1n}+f_{\ 2m})_{n,m\geq 1}$ is a dense sequence of measurable selections of G. Set

$$G_{nm}(\omega) = \{f_{1i}(\omega) + f_{2j}(\omega) : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Then

$$\forall \omega \in \Omega, \lim_{n\to\infty, m\to\infty} d(g(\omega), G_{nm}(\omega)) = 0.$$

There exists a measurable selection g_{nm} of G_{nm} satisfying

$$\|g(\omega) - g_{nm}(\omega)\| = d(g(\omega), G_{nm}(\omega)).$$

Thus $g_{nm}(\omega) \in F_1(\omega) + F_2(\omega)$. Then we obtain measurable selectoins f^i_{nm} of F_i such that $f^1_{nm} + f^2_{nm} = g_{nm}$. This yields

$$d\left(\int_\Omega g_{nm} d\mu, \int_\Omega F_1 d\mu + \int_\Omega F_2 d\mu\right) = 0.$$

By taking the limit when $n, m \to \infty$, the statement ensues.

6. Fix $f_i \in F$, $i = 1, 2$ and $\lambda \in [0, 1]$. We have to prove the existence of $f \in F$ satisfying

$$\lambda \int_\Omega f_1 d\mu + (1-\lambda)\int_\Omega f_2 d\mu = \int_\Omega f d\mu.$$

Define the vector measure $\nu : X \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\nu(A) = (\int_A f_1 d\mu, \int_A f_2 d\mu).$$

Then υ is finite and nonatomic. Lyapunov's Theorem implies that the range of ν is convex and compact. Since

$$\nu(\phi) = \{0\} \ \& \ \nu(\Omega) = (\int\limits_{\Omega} f_1 d\mu, \int\limits_{\Omega} f_2 d\mu)$$

there exists $A \in X$ satisfying $\nu(A) = \lambda\nu(\Omega)$, i.e.,

$$\lambda \int\limits_{\Omega} f_1 d\mu = \int\limits_{A} f_1 d\mu \ , \ \lambda \int\limits_{\Omega} f_2 d\mu = \int\limits_{A} f_2 d\mu.$$

Therefore the map $f = \chi_A f_1 + \chi_{\Omega \setminus A} f_2$ is an integrable selection of F we were looking for. We know that the integral $\int\limits_{\Omega} F d\mu$ contains all extremal points of the convex set $\overline{co} \int\limits_{\Omega} F d\mu$.

Finally, if F is integrably bounded, then the set $\overline{co} \int\limits_{\Omega} F d\mu$ is compact. Since $\int\limits_{\Omega} F d\mu$ is convex and contains all the extremal points of its closed convex hull, we deduce from Caratheodory theorem that

$$\int\limits_{\Omega} F d\mu = co(\int\limits_{\Omega} F d\mu) = \overline{co}(\int\limits_{\Omega} F d\mu).$$

The proof is completed.

7. We have to show that for all $f_i \in F$, $i = 1, 2$, $\varepsilon > 0$ and $\lambda \in [0, 1]$, there exists $f \in F$ satisfying

$$\left\| \lambda \int\limits_{\Omega} f_1 d\mu + (1 - \lambda) \int\limits_{\Omega} f_2 d\mu - \int\limits_{\Omega} f d\mu \right\| \leq \varepsilon.$$

Define the vector measure $\nu : X \to X \times X$ by

$$\nu(A) = (\int\limits_{A} f_1 d\mu, \int\limits_{A} f_2 d\mu).$$

Then the closure of the range of ν is convex. Since

$$\iota)$$

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we have for some $A \in X$

$$\left\|\lambda \int_{\Omega} f_1 d\mu - \int_{A} f_1 d\mu\right\| + \left\|\lambda \int_{\Omega} f_2 d\mu - \int_{A} f_2 d\mu\right\| \leq \varepsilon.$$

Setting $f = \chi_A f_1 + \chi_{\Omega \setminus A} f_2$, we deduce that $\overline{\int_{\Omega} F d\mu}$ is closed and convex.

To prove the last claim, assume that X is reflexive and that F is integrably bounded and has nonempty closed convex images. Then the set F is weakly compact in $\sigma(L^1(\Omega, X, \mu), L^\infty(\Omega, X, \mu))$ and the proof follows. ■

1.3 Fractional Calculus.

Fractional calculus is the branch of calculus that generalizes the derivatives (and integrals) of a function to non-integer order, allowing calculations even making to the number line and the extension of this map to any fractional "differintegrals". Despite "generalized" would be a better option, the name "differintegrals" is used for denoting are kinds of derivative (and integral) which are must include integral transforms. We will use the following definitions and notations, which can be found in [20, 33, 36].

Definition 1.3.1 *(Definition D.1, Theorem D.6 [20]).*

1. The function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \tag{1.3.1}$$

   is called Euler's Gamma function (or Euler's integrals of the second kind).

2. The Beta function (or Euler's integrals of the first kind), defined by

$$\boldsymbol{B}(\beta, \alpha) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

In the following we will mention some of the iterative relations for the Gamma function (see Theorem D.1,[20]):

i. By integration (1.3.1) by parts one checks that for all $\alpha > 0$, we have the recurrence relationship

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha). \tag{1.3.2}$$

Further we have $\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}\,|_0^\infty = 1$.

ii. For all $n \in \mathbb{N} \cup \{0\}, \Gamma(n+1) = n!$.

iii. From the relation (1.3.2) we can define Γ on $(-\infty, 0) - \mathbb{Z}$ by:

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}.$$

For example:

$$\Gamma(-\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi},$$

and

$$\Gamma(-\frac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{(-\frac{1}{2})(-\frac{3}{2})} = \frac{2}{3}\sqrt{\pi}.$$

Definition 1.3.2 ***(Definition 1.1,[20]).*** *Let $J = [a, b]$ be an interval and $n \in \mathbb{N}$,*

1. By D_t, we denote the operator that maps a differentiable function into its derivative, i.e.

$$D_t f(t) = \frac{d}{dt}f(t) = f'(t).$$

2. By ${}_aI_t$, we denote the operator that maps a function f, assumed to be $f \in C[a, b]$, into its primitive centered at a, i.e.

$${}_aI_t f(t) = \int_a^t f(s)ds, \forall t \in [a, b].$$

3. For all $n \in \mathbb{N}$ we use the symbols D_t^n and ${}_aI_t^n$ to denote the n-fold iterates of D_t and ${}_aI_t$, respectively, i.e. , we set $D_t^1 = D_t$, ${}_aI_t^1 = {}_aI_t$, and $D_t^n = D_t[D_t^{n-1}]$and ${}_aI_t^n = {}_aI_t[{}_aI_t^{n-1}]$ for $n \geq 2$. Moreover by fundamental Theorem of classical calculus we have $D_t^n[{}_aI_t^n f(t)] = f(t)$ and ${}_aI_t^n[D_t^n f(t)] \neq f(t)$, for all $n \in \mathbb{N}$ (i.e. D_t^n is a left-inverse not right-inverse) to ${}_aI_t^n$. In fact, ${}_aI_t^n[D_t^n]$ have the expansion

$$ {}_aI_t^n(D_t^n f(t)) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!}, t > a, $$

where, $f \in C^{(k-1)}[a,b]$.

1.3.1 Riemann-Liouville Differential and integral Operators.

In the following we proved the calculation of the n-fold primitive of a function $f(t)$ to a single integral of convolution type.

Lemma 1.3.1.1 (Lemma 1.1.1, [20]**).** Let f be Riemann integrable on $[a,b]$. Then for all $t \in [a,b]$ and $n \in \mathbb{N}$ ($\mathbb{N} = \{1,2,\ldots\}$), the unique solution of the IVB problem

$$ \begin{cases} y^{(n)}(t) = f(t), \\ y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0, \end{cases} $$

is given by Cauchy formula,

$$ \begin{aligned} y(t) &= \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)ds \\ &= \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1} f(s)ds, \forall t \in [a,b]. \end{aligned} $$

Proof.

We will use induction to prove our claim.

For $n = 1$ we have

$$ y'(t) = f(t),\ y(a) = 0. $$

Solving this equation we obtain

$$\int_a^t y'(s)\,ds = \int_a^t \frac{(t-s)^{1-1}}{(1-1)!} f(s)ds.$$

Since $y(a) = 0$, we then have

$$y(t) = \int_a^t f(s)ds.$$

Now we assume that

$$y(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)ds$$

is true for n and show that the equation is also true for $n+1$.

Consider

$$\begin{aligned} y^{(n+1)}(t) &= f(t) \\ y(a) &= y'(a) = \cdots = y^{(n)}(a) = 0. \end{aligned}$$

Since $y^{(n+1)}(t) = (y')^{(n)}(t)$, let $u(t) = y'(t)$. Then

$$\begin{aligned} u^{(n)}(t) &= f(t) \\ u(a) &= u'(a) = \cdots = u^{(n-1)}(a) = 0. \end{aligned}$$

Using the induction hypothesis we observe that,

$$\int_a^t y'(s)\,ds = \int_{z=a}^t \left(\int_{a=s}^z \frac{(z-s)^{n-1}}{(n-1)!} f(s)ds\right) dz$$

$$\begin{aligned} y(t) - y(a) &= \int_{a=s}^t \left(\int_{z=s}^t \frac{(z-s)^{n-1}}{(n-1)!} f(s)dz\right) ds \\ &= \int_a^t \frac{(t-s)^n}{n!} f(s)ds. \end{aligned}$$

Since $y(a) = 0$, then

$$y(t) = \int_a^t \frac{(t-s)^n}{n!} f(s)ds.$$

Note that, since $f(t)$ is the n^{th} derivative of $y(t)$ and $y(a) = y'(a) = \cdots = y^{(n)}(a) = 0$, we may interpret $y(t)$ as the n^{th} integral of $f(t)$. Thus,

$$ {}_aI_t^n f(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)ds. $$

Finally, if we change the fractional into Gamma function, then

$$ \begin{aligned} {}_aI_t^n f(t) &= \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)ds \\ &= \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1} f(s)ds, \end{aligned} $$

the Riemann- Liouville definition of a fractional integral. ■

The existence of the integral ${}_aI_t^n$ is obvious when $n \geq 1$. In the case $0 < n < 1$ though, the situation is less clear at first sight. However, the following result asserts existence ${}_aI_t^n$ also.

Theorem 1.3.1.1 (Th 2.2.1, [20] **).** Let $f \in L^1[a,b]$ and $n > 0$. Then, the integral ${}_aI_t^n f(t)$ exists for all $a.e.$ $t \in [a,b]$. Moreover, the function ${}_aI_t^n f(t) \in L^1[a,b]$.

Proof.

We write the integral as:

$$ \int_a^t (t-s)^{n-1} f(s)ds = \int_{-\infty}^{\infty} \phi_1(t-s)\phi_2(t)ds $$

where

$$ \begin{aligned} \phi_1(u) &= \begin{cases} u^{n-1} & \text{for} \quad 0 < u \leq b-a, \\ 0 & \text{else}, \end{cases} \\ \phi_2(u) &= \begin{cases} f(u) & \text{for} \quad a \leq u \leq b, \\ 0 & \text{else}. \end{cases} \end{aligned} $$

By construction, $\phi_j \in L^1(\mathbb{R})$ for $j \in \{1,2\}$, and thus by a classical result on Lebesgue integration the desired result follows. ■

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Definition 1.3.1.1 **(**Definition 2.1,[20]**,** Lemma 2.2,[44]**).** Let X be a Banach space and $q > 0$. The operator ${}_aI_t^q$ is defined on $L^1([a,b], X)$, by

$$ {}_aI_t^q f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s) ds, $$

for all $t \in [a,b]$ is called the Riemann-Liouville integrals operator of order $q$and with lower limit a. For $q = 0$, we set ${}_aI_t^0 = I$, the identity operator. When $a = 0$, we write ${}_0I_t^q f(t) = f(t) * \varphi_q(t)$, where

$$ \varphi_q(t) = \begin{cases} \frac{t^{q-1}}{\Gamma(q)}, & \forall t > 0, \\ 0, & \forall t \leq 0. \end{cases} $$

Theorem 1.3.1.2 **(**Th 2.2.7, Th 2.2.9, [20]**).** Let $n > 0$, $m > 0$, then the fractional integrals have the following properties,

1. Interchange with limit operation, that is if $(f_k)_{k \geq 1} \in C[a,b]$, is uniformly convergent sequence. Then

$$ \left({}_aI_t^n \lim_{k \to \infty} f_k \right)(t) = \left(\lim_{k \to \infty} \left({}_aI_t^n \right) f_k \right)(t). $$

In particular, the sequence of functions $\left({}_aI_t^n f_k \right)_{k \geq 1}$is uniformly convergent.

2. The continuity with respect to the fractional integrals, that is if $1 \leq p < \infty$ and $(m_k)_{k \geq 1}$be a convergent sequence of nonnegative numbers with limit m. Then, for all $f \in L^p[a,b]$,

$$ \lim_{k \to \infty} \| {}_aI_t^{m_k} f - {}_aI_t^m f \|_\infty = \lim_{k \to \infty} \sup_{t \in [a,b]} | {}_aI_t^{m_k} f - {}_aI_t^m f | = 0, $$

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Proof. 1. We denote the limit of sequence (f_k) by f. It is well known that f is continuous. We then find

$$\begin{aligned} \left| {}_aI_t^n f_k(t) - {}_aI_t^n f(t) \right| &\leq \frac{1}{\Gamma(n)} \int_a^t |f_k(s) - f(s)| (t-s)^{n-1} \, ds \\ &\leq \frac{1}{\Gamma(n)} \|f_k - f\|_\infty \int_a^t (t-s)^{n-1} ds \\ &= \frac{1}{\Gamma(n+1)} \|f_k - f\|_\infty (t-a)^n \\ &\leq \frac{1}{\Gamma(n+1)} \|f_k - f\|_\infty (b-a)^n \end{aligned}$$

which converges to zero as $k \to \infty$ uniformly for all $t \in [a, b]$.

To prove (2), we refer the reader to([20], Theorem 2.2.9). ■

Remark 1.3.1.1 (Example 2.3,[20]).

In assertion 2, if $m = 0$ then the sequence $(m_k)_{k \geq 1}$ must be decreasing. Moreover if $f(t) = 1$, we have that

$$ {}_aI_t^{m_k} f(t) = \frac{(t-a)^{m_k}}{\Gamma(m_k+1)}, $$

so, ${}_aI_t^{m_k} f(a) = 0$, for all k whereas

$$ {}_aI_t^m f(a) = {}_aI_t^0 f(a) = f(a) = 1, $$

i.e. we do not even have pointwise convergence. So to occur the uniform convergence when $m = 0$, $f(t)$ must be take the form $O((t-a)^\delta)$ as $t \to \delta$ for some $\delta > 0$.

One important property of integer-order integral operators is preserved by our generalization:

Theorem 1.3.1.3 (Theorem 2.2, Corollary 2.4, [20]). Let $m, n \geq 0$ and $\varphi \in L^1[a, b]$. Then,

$$ {}_aI_t^m [{}_aI_t^n \varphi(t)] = I_t^{m+n} \varphi, $$

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holds almost everywhere on $[a,b]$. If additionally $\varphi \in C[a,b]$ or $m+n \geq 1$, then the identity holds everywhere on $[a,b]$. Moreover

$$ {}_aI_t^m[{}_aI_t^n\varphi(t)] = {}_aI_t^n[{}_aI_t^m\varphi(t)]. $$

Proof.

We have

$$ {}_aI_t^m[{}_aI_t^n\varphi(t)] = \frac{1}{\Gamma(m)\Gamma(n)}\int_a^t (t-s)^{m-1}\int_a^s (s-\mu)^{n-1}\phi(\mu)\,d\mu ds. $$

In view of Theorem (1.3.1.1), the integrals exist, and by Fubini's theorem we may interchange the order of integration, obtaining

$$ \begin{aligned} {}_aI_t^m[{}_aI_t^n\varphi(t)] &= \frac{1}{\Gamma(m)\Gamma(n)}\int_a^t\int_\mu^t (t-s)^{m-1}(s-\mu)^{n-1}\phi(\mu)\,dsd\mu \\ &= \frac{1}{\Gamma(m)\Gamma(n)}\int_a^t \phi(\mu)\int_\mu^t (t-s)^{m-1}(s-\mu)^{n-1}dsd\mu. \end{aligned} $$

The substitution $s = \mu + k(t-\mu)$ yields

$$ \begin{aligned} {}_aI_t^m[{}_aI_t^n\varphi(t)] &= \frac{1}{\Gamma(m)\Gamma(n)}\int_a^t \phi(\mu)\int_0^1 [(t-\mu)(1-k)]^{m-1} \times [k(t-\mu)^{n-1}(t-\mu)]dkd\mu \\ &= \frac{1}{\Gamma(m)\Gamma(n)}\int_a^t \phi(\mu)\,(t-\mu)^{m+n-1}\int_0^1 (1-k)^{m-1}k^{n-1}dkd\mu. \end{aligned} $$

We know that $\int_0^1 (1-k)^{m-1}k^{n-1}dk = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, and thus

$$ {}_aI_t^m[{}_aI_t^n\varphi(t)] = \frac{1}{\Gamma(m+n)}\int_a^t \phi(\mu)\,(t-\mu)^{m+n-1}d\mu = {}_aI_t^{m+n}\varphi(t) $$

almost everywhere on $[a,b]$.

Moreover, by the classical theorems on parameter integrals, if $\varphi \in C[a,b]$ then also ${}_aI_t^n\varphi(t) \in C[a,b]$, and therefore ${}_aI_t^m[{}_aI_t^n\varphi(t)] \in C[a,b]$, and ${}_aI_t^{m+n}\varphi(t) \in C[a,b]$ too. Thus, since these two continuous functions coincide almost everywhere, they must coincide everywhere.

Finally, if $\varphi \in L^1[a,b]$ and $m+n \geq 1$ we have, by the result above

$$ {}_aI_t^m[{}_aI_t^n\varphi(t)] = {}_aI_t^{m+n}\varphi(t) = {}_aI_t^{m+n-1}\ {}_aI_t^1\varphi(t) $$

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almost everywhere. Since ${}_aI_t^1\varphi(t)$ is continuous, we also have that ${}_aI_t^{m+n}\varphi(t) = {}_aI_t^{m+n-1}\ {}_aI_t^1\varphi(t)$ is continuous, and once again we may conclude that the two functions on either side of the equality almost everywhere are continuous; thus they must be identical everywhere. ■

Example 1.3.1.1 (Example 1.3,[27], Example 2.2,[20]).

1. Let $f(t) = (t-a)^\beta$, $\beta > -1$ and $q > 0$. Then

$$\begin{aligned}{}_aI_t^q f(t) &= \frac{1}{\Gamma(q)}\int_a^t (s-a)^\beta (t-s)^{q-1} ds \\ &= \frac{1}{\Gamma(q)}(t-a)^{q+\beta}\int_0^1 y^\beta (1-y)^{q-1} dy \\ &= \frac{\Gamma(\beta+1)}{\Gamma(q+\beta+1)}(t-a)^{q+\beta}.\end{aligned}$$

   This result is precisely is a generalization of the integral operator when $q \in \mathbb{N}$.

2. Let $f(t) = \exp(\lambda t)$ with some $\lambda > 0$. Compute ${}_0I_t^n f(t)$ for $n > 0$. In the case $n \in \mathbb{N}$ we obviously have

$${}_0I_t^n f(t) = \lambda^{-n}\exp(\lambda t).$$

   However, this result does not generalize in a straightforward way to the case $n \notin \mathbb{N}$. Rather, in view of the well knows series expansion of the exponential function, Theorem (1.3.1.2), and the last example, we find

$$\begin{aligned}{}_0I_t^n f(t) &= {}_0I_t^n\left[\sum_{k=0}^{\infty}\frac{(\lambda\cdot)^k}{k!}\right](t) = \sum_{k=0}^{\infty}(\frac{(\lambda\cdot)^k}{k!}){}_0I_t^n\left[(\cdot)^k\right](t) \\ &= \sum_{k=0}^{\infty}\frac{(\lambda)^k}{\Gamma(k+n+1)}t^{k+n} = \lambda^{-n}\sum_{k=0}^{\infty}\frac{(\lambda t)^{k+n}}{\Gamma(k+n+1)},\end{aligned}$$

   and here the series on the right-hand side is not $\exp(\lambda t)$.

Definition 1.3.1.2 (Definition 2.2,[20], Lemma 2.2,[44]). Let X be a Banach space, $q > 0$ and $m = \lceil q \rceil$ (denote the smallest integer greater than or equal q) and

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$f \in L^1([a,b], X)$. The operator ${}_aD_t^q$, defined by

$$ {}_a^L D_t^q f = D_t^m \left[{}_aI_t^{m-q} f \right] = \begin{cases} \dfrac{1}{\Gamma(m-q)} (\dfrac{d}{dt})^m \displaystyle\int_a^t (t-s)^{m-q-1} f(s) ds & \text{if } q \notin \mathbb{N}, \\ D_t^m f(t) \text{ if } q \in \mathbb{N}, \end{cases} \tag{1.3.3} $$

is called the Riemann-Liouville fractional differential operator of order q. For $n = 0$, we set

$$ {}_a^L D_t^0 = I, $$

the identity operator, then we easily recognize that

$$ {}_a^L D_t^q [{}_aI_t^q] = I, \forall q \geq 0. $$

The next result contains a very simple sufficient condition for the existence of ${}_aD_t^q f$.

Proposition 1.3.1.1 (Proposition 1.1.1, [27]).

a. If $f \in AC[a,b]$, then ${}_a^L D_t^q f$ exists $a.e.$ for all $0 < q < 1$. Moreover ${}_a^L D_t^q f \in L^p[a,b]$ for $1 \leq p < \dfrac{1}{q}$ with

$$ {}_a^L D_t^q f(t) = \frac{1}{\Gamma(1-q)} \left\{ f(a) (t-a)^{-q} + \int_a^t f'(s) (t-s)^{-q} ds \right\}. $$

b. If $f \in AC^{n-1}[a,b]$, $n = \lceil q \rceil$, then ${}_t^L D_a^q f$ exists $a.e.$, for $q \geq 0$ and has the representation

$$ {}_a^L D_t^q f(t) = \sum_{k=0}^{n-1} \frac{f^k(a)}{\Gamma(1+k-q)} (t-a)^{k-q} + \frac{1}{\Gamma(n-q)} \int_a^t f^{(n)}(s) (t-s)^{n-q-1} ds. \tag{1.3.4} $$

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Proof.

a. We use the definition of the Riemann-Liouville differential operator and the fact that $f \in AC[a,b]$. This yields

$$\begin{aligned} {}_a^L D_t^q f(t) &= \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-q} ds \\ &= \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t \left(f(a) + \int_a^s f'(u) du \right) (t-s)^{-q} ds \\ &= \frac{1}{\Gamma(1-q)} \frac{d}{dt} \left(f(a) \int_a^t \frac{ds}{(t-s)^q} + \int_a^t \int_a^s f'(u)(t-s)^{-q} du\, ds \right) \\ &= \frac{1}{\Gamma(1-q)} \left(\frac{f(a)}{(t-a)^q} + \frac{d}{dt} \int_a^t \int_a^s f'(u)(t-s)^{-q} du\, ds \right). \end{aligned}$$

By the Fubini's Theorem we may interchange the order of integration in the double integral. This yields

$$ {}_a^L D_t^q f(t) = \frac{1}{\Gamma(1-q)} \left(\frac{f(a)}{(t-a)^q} + \frac{d}{dt} \int_a^t f'(u) \frac{(t-u)^{1-q}}{1-q} du \right).$$

The standard rules on the differentiation of parameter integrals then give the desired representation. The integrability statement is immediate consequence of this representation using classical result from Lebesgue integration theory.

b. By definition, ${}_a^L D_t^q f(t) = {}_a^L D_t^n \, {}_a I_t^{n-q} f(t)$ and

$$ {}_a I_t^{n-q} f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} f(s) ds.$$

In view of the smoothness assumptions on f, we may integrate partially in this integral and find that

$$\begin{aligned} {}_a I_t^{n-q} f(t) &= \frac{1}{\Gamma(n-q+1)} (t-s)^{n-q} f(s)|_{s=a}^{s=t} - \int_a^t \frac{1}{\Gamma(n-q+1)} (t-s)^{n-q} f'(s) \\ &= \frac{1}{\Gamma(n-q+1)} (t-a)^{n-q} f(a) + {}_a I_t^{n-q+1} f'(t). \end{aligned}$$

The smoothness assumptions allow us to repeat this procedure for a total of n times; we find

$$ {}_a I_t^{n-q} f(t) = \sum_{k=0}^{n-1} \frac{(t-a)^{k+n-q}}{\Gamma(k+n-q+1)} f^{(k)}(a) + {}_a I_t^{2n-q} f^{(n)}(t).$$

Thus,

$${}_a^L D_t^q f(t) = {}_a^L D_t^n \; {}_a I_t^{n-q} f(t) = \sum_{k=0}^{n-1} \frac{(t-a)^{k-q}}{\Gamma(k-q+1)} f^{(k)}(a) + {}_a I_t^{n-q} f^{(n)}(t).$$

But the expression on the right-hand side of the equation is just

$$\int_a^t \frac{(t-s)^{-q-1}}{\Gamma(-q)} f(s)ds.$$

■

Example 1.3.1.2 (Example 1.3,[27]).

Compute ${}_0^L D_t^q \; t^\beta, \beta > -1$ and $q > 0$. Let $\lceil q \rceil = m$. We have

$$\begin{aligned}
{}_0^L D_t^q \; t^\beta &= \left(\frac{d}{dt}\right)^m \left\{ \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} s^\beta ds \right\} \\
&= \frac{1}{\Gamma(m-q)} \left(\frac{d}{dt}\right)^m \left\{ t^{m-q+\beta} \int_0^1 (1-u)^{m-q-1} u^\beta du \right\} \\
&= \frac{1}{\Gamma(m-q)} \frac{\Gamma(\beta+1)\Gamma(m-q)}{\Gamma(\beta+1+m-q)} \left(\frac{d}{dt}\right)^m t^{m-q+\beta} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-q)} t^{\beta-q} \frac{\Gamma(\beta+1+m-q)}{\Gamma(\beta-m+m-q+1)} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-q+1)} t^{\beta-q}. \qquad (1.3.5)
\end{aligned}$$

Note that, due to the poles of the Gamma function at the points $0.-1, -2, -3, ...$,we obtain

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta-q+1)} = \begin{cases} \beta(\beta-1)\cdots(\beta-q+1), & \beta-q \notin \{-1,-2,-3,...\}, \in \mathbb{N}. \\ 0, \quad \text{if} & \beta-q \in \{-1,-2,-3,...\}. \\ \dfrac{1}{\Gamma(1-q)} & ,\beta=0, q \notin \mathbb{N} \end{cases}$$

Theorem 1.3.1.4 (Properties 2.2 and 2.3, [33]). Let $q > 0$, $\beta < q$ and $f \in L^1[a,b]$.

Then

(1) ${}_a D_t^\beta \; {}_a I_t^q f(t) = {}_a I_t^{q-\beta} f(t)$, $a.e.$

In particular, when $\beta = k \in \mathbb{N}$ and $k < q$, then

$$ {}_aD_t^k \, {}_aI_t^\alpha f(t) = {}_aI_t^{\alpha-k} f(t), \; a.e. $$

(2) If the fractional derivatives ${}_aD_t^q f(t)$ and ${}_aD_t^{q+k} f(t)$ exist, $k \in \mathbb{N}$, then ${}_aD_t^k$ ${}_aD_t^q f(t) = {}_aD_t^{q+k} f(t)$, $\forall t \in [a, b]$.

Theorem 1.3.1.5 ([20], **Theorem 2.14, Theorem 2.22**)

Let $q \geq 0$. Then

(1) for every $f \in L^1[a, b]$,

$$ {}_a^LD_t^q[{}_aI_t^q f] = f, a.e., $$

so ${}_a^LD_t^q$ is left inverse of ${}_aI_t^q f$.

(2) If $f \in L^1[a, b]$ and there is $\varphi \in L^1[a, b]$ such that ${}_aI_t^q \varphi = f$ then

$$ {}_aI_t^q[{}_a^LD_t^q f] = f, a.e. $$

The following example shows that ${}_a^LD_t^q$ is not right inverse to ${}_a^LD_t^q$ in general.

Example 1.3.1.3.

Let

$$ f(t) = t^{q-1}, q > 0, t > 0. $$

From examples (1.3.1.1) and (1.3.1.2) we have ${}_0^LD_t^q t^{q-1} = 0$. Then ${}_0I_t^q({}_0^LD_t^q t^{q-1}) = 0$, but

$$ \begin{aligned} {}_0^LD_t^q[{}_0I_t^q t^{q-1}] &= {}_0^LD_t^q(\frac{\Gamma(q)}{\Gamma(2q)} t^{2q-1}) \\ &= \frac{\Gamma(q)}{\Gamma(2q)} \, {}_0^LD_t^q t^{2q-1} = \frac{\Gamma(q)}{\Gamma(2q)} \frac{\Gamma(2q)}{\Gamma(q)} t^{q-1} \\ &= t^{q-1}. \end{aligned} $$

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1.3.2 Caputo fractional derivative.

An alternative way to define a fractional derivative of order q, originally introduced by Caputo in the late sixties and adopted by Caputo and Mainardi [13] in the framework of the theory of Linear Viscoelasticity.

Definition 1.3.2.1 **([20,23]).** Let $q > 0$and m be an integer such that $m-1 < q < m$, and X be a Banach space. *The Caputo derivative of order* q with the lower limit a *for a given function* $f \in L^1([a,b],X)$ *is* defined by

$$ {}_a^cD_t^q f(t) = {}_a^LD_t^q \left(f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right). \tag{1.3.6}$$

provided the right side is point-wise defined on J.

It is known that

(i) If $f \in C^m\ ([a,b],E)$, then

$$ {}_a^cD_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^{(n)}(s)ds. \tag{1.3.7}$$

(ii) If $f \in L^p([a,b],E), P > \frac{1}{q}$,then $(I^q f)^{(k)}(t), k = 1,2,..,m-1$ exists at any $t \in J$, and $(I^q f)^{(k)}(a) = 0, k = 1,2,..,m-1$. So

$$ {}_a^cD_t^q\ ({}_aI^q f(t)) = {}_a^LD_t^q\ ({}_aI^q f(t)) = f(t), a.e.t \in J.$$

By example (1.3.1.2) , the relation between the two fractional derivatives (1.3.3) and (1.3.6) is given by

$$ {}_a^LD_t^q f(t) = {}_a^cD_t^q f(t) + \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(1+k-q)} (t-a)^{k-q}. \tag{1.3.8}$$

Note that

$$ {}_a^LD_t^q f(t) = {}_a^cD_t^q f(t),$$

if and only if

$$ D_t^k f(a^+) = 0, \forall k = 0,1,\ldots,m-1. \tag{1.3.9}$$

Example 1.3.2.1 (Example 3.1,[20]).

Let $q>0, m=\lceil q\rceil$, and $f(t)=(t-a)^{\beta}$ for some $\beta \geq 0$. Then,

$$ {}_{a}^{c}D_{t}^{q}f(t)=\begin{cases} 0, & \text{if} \quad \beta \in \{0,1,2,\ldots,m-1\}, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-q)}(t-a)^{\beta-q}, & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq m, \text{or} \quad \beta \notin \mathbb{N} \text{ and } \quad \beta > m-1. \end{cases} $$

If we compare this statement with the corresponding one for Riemann-Liouville operators (Example 1.3.1.2) to get the two operators have different kernels, and that the domain of the two operators are also different

When it comes to the composition of Riemann-Liouville integrals and Caputo differential operators, we find that the Caputo is also a left inverse of the Riemann-Liouville integral.

Theorem 1.3.2.1 (Lemma 2.21, [33]). Let $q>0$ and $f \in L^{\infty}[a,b]$, then

$$ {}_{a}^{c}D_{t}^{q}({}_{a}I_{t}^{q}f(t))=f(t), \ \forall t \in [a,b] $$

Proof.

Let $m=\lceil q\rceil$. By Theorem (1.3.1.4), we have

$$ ({}_{a}I_{t}^{q}f(t))^{(k)}= \ {}_{a}I_{t}^{q-k}f(t), \ \forall k=0,1,....,m-1. $$

From Hölder inequality we get

$$ \begin{aligned} \left|{}_{a}I_{t}^{q-k}f(t)\right| &\leq \frac{1}{\Gamma(q-k)}\int_{a}^{t}(t-s)^{q-k-1}\left|f(s)\right|ds \\ &\leq \|f\|_{\infty}\frac{1}{\Gamma(q-k)}\int_{a}^{t}(t-s)^{q-k-1}ds \\ &\leq \|f\|_{\infty}\frac{1}{\Gamma(q-k)}\frac{(t-s)^{q-k}}{q-k}\Big|_{a}^{t} \\ &= \|f\|_{\infty}\frac{1}{\Gamma(q-k)}\frac{(t-a)^{q-k}}{q-k} \end{aligned} $$

so, ${}_{a}I_{t}^{q-k}f(t)=0$ for $k=0,1,\ldots,m-1$.

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Thus in view of (1.3.7) and (1.3.9) with the property of ${}_a^L D_t^q$ that is a left inverse of ${}_a I_t^q$

$$ {}_a^c D_t^q ({}_a I_t^q \ f) = {}_a^L D_t^q \ {}_a I_t^q f = f. $$

Once a gain, we find that the Caputo derivative is not the right inverse of the Riemann-Liouville integral. ■

Lemma 1.3.2.1 **(**Th 3.3.8, [20]**)** Assume that $n \geq 0$, $m = \lceil n \rceil$ and $f \in AC^m [a, b]$.

Then

$$ {}_a I_t^q ({}_a^c D_t^q f (t)) = f (t) - \sum_{k=0}^{m-1} \frac{D_t^{(k)} f(a)}{k!} (t-a)^k . $$

Proof.

By Definition (1.3.2.1); we have

$$ {}_a^c D_t^q f (t) = {}_a I_t^{m-q} ({}_a^c D_t^m f (t)). $$

Thus, applying the operator ${}_a I_t^q$ to both sides of this equation and using the semi-group property of fractional integration, we obtain

$$ {}_a I_t^q \ {}_a^c D_t^q f (t) = {}_a I_t^q \ {}_a I_t^{m-q} ({}_a^c D_t^m f (t)) = {}_a I_t^m ({}_a^c D_t^m f (t)). $$

By the classical version of Taylor's theorem, we have that

$$ f(t) = \sum_{k=0}^{m-1} \frac{D_t^{(k)} f(a)}{k!} (t-a)^k + {}_a I_t^m ({}_a^c D_t^m f (t)). $$

Combinig these two equations we derive the claim. ■

1.4 Semigroups of Linear Operators.

The theory of strongly continuous semigroup of linear operators on Banach space, operator semigroup, for short, has becomes an indispensable tool in a great number of areas of modern mathematical analysis. It started in the first half of this century,

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acquired its core in 1948 with the Hille-Yosida generating Theorem. However, in general semigroups can be used to solve a large class of problems, for instance, they are usually described by an initial value problem (IVP). In this section we collect the basic notions and facts of strongly semigroups and special class of it.

1.4.1 Uniformly continuous semigroups of bounded linear operators.

Let X be a Banach space and $\pounds(X)$ be the set of all linear bounded operators from X into X. Endowed with the norm $\|\cdot\|_{\pounds(X)}$ defined by

$$\|U\|_{\pounds(X)} = \sup_{\|x\|\leq 1} \|Ux\|,$$

in $\pounds(X)$.

Definition 1.4.1.1 (Definition 2.1.1,[41])**.** A one parameter family $\{U(t), t \geq 0\}$ in $\pounds(X)$, is called a "semigroup of bounded linear operators" on X or simply semigroup if it satisfies the functional equation

$$\begin{cases} U(0) = I, & (I \text{ is the identity operator } X) \\ U(t+s) = U(t)U(s), \ \forall t, s \geq 0 & (\text{the semigroup property}). \end{cases} \quad (1.4.1)$$

If in addition, it satisfies the continuity condition at $t = 0$,

$$\lim_{t\downarrow 0} \|U(t) - I\|_{\pounds(X)} = 0,$$

the semigroup is called "uniformly continuous".

Definition 1.4.1.2 (Definition 1.1.1.[39])**.** The linear operator A defined by

$$Ax = \lim_{t\downarrow 0} \frac{U(t)x - x}{t} = \left.\frac{d^{\,+}U(t)x}{dt}\right|_{t=0} \text{ for } x \in D(A),$$

is called the infinitesimal generator of the semigroup $U(t)$, where $D(A)$ is the domain of A given by the formula

$$D(A) = \left\{x \in X : \lim_{t\downarrow 0} \frac{U(t)x - x}{t} \text{ exists}\right\}.$$

Remark 1.4.1.1 **(**Remark 2.1.1,[41]**).**

1. If A is the infinitesimal generator of the semigroup of linear operators then $D(A)$ is a vector subspace of X and A is a possibly unbounded linear operator.

2. For $A \in \pounds(X)$ let
$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, t \in \mathbb{R}^+,$$
where the convergence of this series takes place in the Banach algebra $\pounds(X)$, thus e^{tA} is a well defined bounded operator on X.

Theorem 1.4.1.1 **(**Th 1.3, [39]**).** Let $U(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If
$$\lim_{t\downarrow 0} \frac{U(t)-I}{t} = A = \lim_{t\downarrow 0} \frac{S(t)-I}{t},$$
then $U(t) = S(t)$ for $t \geq 0$.

Proof.

We will show that given $U > 0$, $S(t) = U(t)$ for $0 \leq t \leq U$. Let $U > 0$ be fixed, since $t \to \|U(t)\|$ and $t \to \|S(t)\|$ are continuous there is a constant C such that $\|U(t)\| \|S(s)\| \leq C$ for $0 \leq s, t \leq U$. Given $\varepsilon > 0$ it follows from $\lim_{t\downarrow 0} \frac{U(t)-I}{t} = A = \lim_{t\downarrow 0} \frac{S(t)-I}{t}$ that there is a $\delta > 0$ such that
$$h^{-1} \|U(h) - S(h)\| < \frac{\varepsilon}{UC} \text{ for } 0 \leq h \leq \delta.$$
Let $0 \leq t \leq U$ and choose $n \geq 1$ such that $\frac{t}{n} < \delta$. From the semigroup property and last equation it then follows that
$$\begin{aligned} \|U(t) - S(t)\| &= \left\| U\left(n\frac{t}{n}\right) - S\left(n\frac{t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| U\left((n-k)\frac{t}{n}\right) S\left(\frac{kt}{n}\right) - U\left((n-k-1)\frac{t}{n}\right) S\left(\frac{(k+1)t}{n}\right) \right\| \\ &\leq \sum_{k=0}^{n-1} \left\| U\left((n-k-1)\frac{t}{n}\right) \right\| \left\| U\left(\frac{t}{n}\right) - S\left(\frac{t}{n}\right) \right\| \left\| S\left(\frac{kt}{n}\right) \right\| \leq Cn\frac{\varepsilon}{UC}\frac{t}{n} \leq \varepsilon. \end{aligned}$$

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Since $\varepsilon > 0$ was arbitrary $U(t) = S(t)$ for $0 \le t \le U$ and the proof is complete. ■

Corollary 1.4.1.1 (Cor 2.1.1, Proposition 2.1.1, [41]**,** Cor.1.4, [39]**).** For $A \in \pounds(X)$, Let $U(t)$ be an uniformly continuous semigroup of bounded linear operators. Then the following properties hold.

1. For all $t \geq 0$, $U(t)$ is invertible,
2. The mapping $t \to U(t)$ is continuous from $[0, \infty)$ to $\pounds(X)$ endowed with the operator norm,
3. There exists a constant $\omega \geq 0$ such that $\|U(t)\| \leq e^{\omega t}$,
4. There exists a unique bounded linear operator A such that $U(t) = e^{tA}$, where the operator A is the infinitesimal generator of $U(t)$,
5. The mapping $t \to U(t)$ is differentiable in norm and satisfies the differential equation
$$\frac{d\,U(t)}{dt} = AU(t) = U(t)A.$$

Proof. 1. Inasmuch as

$$\lim_{t \downarrow 0} U(t) - I = 0,$$

in the norm topology of $\pounds(X)$, there exists $\delta > 0$ such that

$$\|U(t) - I\|_{\pounds(X)} < 1$$

for each $t \in (0, \delta]$. Thus, for each $t \in (0, \delta]$, $U(t)$ is invertible. Let $t > \delta$. Then there exist $n \in \mathbb{N}^*$ and $\eta \in [0, \delta)$ such that $t = n\delta + \eta$. Therefore $U(t) = U(\delta)^n U(\eta)$, and so $U(t)$ is invertible. The proof is complete.

2. Let $\{G(t); t \in \mathbb{R}\}$ be the uniformly continuous group of linear operators which extends $\{U(t); t \geq 0\}$ and let $t > 0$. Then

$$\lim_{h \to 0} \|U(t+h) - U(t)\|_{\pounds(X)} \leq \lim_{h \to 0} \|U(t)\|_{\pounds(X)} \|G(h) - I\|_{\pounds(X)} = 0.$$

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As at $t = 0$ the continuity follows from Definition (1.4.1.1), this achieves the proof.

The last three properties follow easily from the fourth one.

To prove (4) note that the infinitesimal generator of $U(t)$ is a bounded linear operator A. A is also the infinitesimal generator of e^{tA} defined by $U(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$ and therefore, by Theorem (1.4.1.1), $U(t) = e^{tA}$. ■

1.4.2 Strongly continuous semigroup of bounded linear operators.

Definition 1.4.2.1 **(**Definition 2.3.1,[41]**).** A family $\{U(t), t \geq 0\}$ of bounded linear operators on X, is called a strongly continuous (one-parameter) semigroup or semigroup of class C_0 or ' C_0-semigroup' if it satisfies the functional equation (1.4.1) and is strongly continuous in the following sense

$$\lim_{t \downarrow 0} U(t)x = x \quad \text{for every } x \in X. \tag{1.4.2}$$

Finally, if these properties hold for $\mathbb{R}$ instead of $\mathbb{R}^+$, we call $\{U(t), t \in \mathbb{R}\}$ a strongly continuous (one-parameter) group or (C_0-semigroup) on X.

Example 1.4.2.1 **(**Example 2.3.1,[41]**).**

Let X be the Banach space of bounded uniformly continuous functions on $\mathbb{R}$ with the supremum norm. For $f \in X$ we define the left translation semigroup

$$(U(t)f)(s) = f(t+s).$$

It is easy to check that $U(t)$ is a C_0- semigroup satisfying $\|U(t)\| \leq 1$ for $t \geq 0$. The infinitesimal generator of $U(t)$ is defined on $D(A) = \{f : f \in X,\ f' \text{ exists}, f' \in X\}$ and

$$\begin{aligned}
(Af)(s) &= \lim_{t \downarrow 0} \frac{(U(t)f)(s) - f(s)}{t} \\
&= \lim_{t \downarrow 0} \frac{f(t+s) - f(s)}{t} = f'(s),
\end{aligned}$$

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for $f \in D(A)$.

Remark 1.4.2.1 (Remark 2.3.2, Theorem 2.4.2,[41]).

1. Each uniformly continuous group is of class C_0. The converse is not true in general as we can state from the Example (1.4.2.1), and take for all $t \geq 0$,

$$f(\tau) = \begin{cases} 1 - 2\frac{\tau}{t}, & \text{if } 0 \leq \tau \leq \frac{t}{2}, \\ 0, & \text{if } \quad \tau > \frac{t}{2}. \end{cases}$$

2. The Theorem (1.4.1.1) is true even on a C_0- semigroup.

3. We repeat that for a strongly continuous semigroup $\{U(t), t \geq 0\}$ the finite orbits $\{U(t), t \in [0, t_0]\}$ are continuous images of a compact interval, hence compact and therefore bounded for all $x \in X$. So by the uniform boundedness principles each strongly continuous semigroup is uniformly bounded on each compact interval, a fact that implies exponential boundedness on $\mathbb{R}^+$.

Theorem 1.4.2.1 (Theorem2.3.1, [41]). Let $U(t)$ be a C_0- semigroup. Then there exists constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|U(t)\| \leq Me^{\omega t}, \forall\ t \geq 0$.

Proof.

First we show that $\|U(t)\|$ is bounded on $0 \leq t \leq \eta$ for some $\eta > 0$. If not then there exist $\{t_n\}$ such that $t_n \downarrow 0$ and $\|U(t_n)\| \geq n$. From the principle of uniform boundedness there must exist a $x \in X$ such that $\|U(t_n)x\|$ is unbounded. But this contradicts the strong continuity at $t = 0$. Thus there exists η and M such that $\|U(t)\| \leq M$ for $0 \leq t \leq \eta$. Also since $\|U(0)\| = 1$ we must have $M \geq 1$.

Let $\omega = \frac{\ln(M)}{\eta}$. Given any $t \geq 0$ there exists $n \in \mathbb{Z}$ and δ with $0 \leq \delta \leq \eta$ so that $t = n\eta + \delta$. Then by the semigroup property we have

$$\|U(t)\| = \|U(\delta)U(\eta)^n\| \leq M^{n+1}.$$

Now $t = n\eta + \delta$ implies

$$n = \frac{t-\delta}{\eta} \leq \frac{t}{\eta}$$

and since $M \geq 1$ we have

$$\|U(t)\| \leq MM^n \leq MM^{\frac{t}{\eta}}.$$

Now $\omega = \frac{\ln M}{\eta}$ implies $\ln M = \eta\omega$ which implies $M = e^{\omega\eta}$. Thus we have

$$\|U(t)\| \leq MM^{\frac{t}{\eta}} \leq M(e^{\omega\eta})^{\frac{t}{\eta}} \leq Me^{\omega t}.$$

■

Definition 1.4.2.2 **(**Definition 2.3.2,[41]**).** A C_0- semigroup $\{U(t), t \geq 0\}$ is called of type (M, ω) with $M \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have

$$\|U(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}.$$

Moreover a semigroup is called bounded if we can take $\omega = 0$ and contractions, or of nonexpansive operators, if it is of type $(1, 0)$, i.e., if for each $t \geq 0$, we have

$$\|U(t)\|_{\mathcal{L}(X)} \leq 1.$$

The number

$$\omega_0(U) = \inf\left\{\omega \in \mathbb{R} : \exists M(\omega) \geq 1, \|U(t)\| \leq Me^{\omega t}, t \geq 0\right\},$$

is called the growth bounded of U.

Definition 1.4.2.3 **(**Definition 2.4.1,[41]**).** An operator $A : D(A) \subseteq X \to X$ is closed in $X \times X$, that is, for all $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ in X, then $x \in D(A)$ and $Ax = y$.

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Corollary 1.4.2.1 (Cor 1.2.5, [39]). If A is the infinitesimal generator of a C_0-semigroup $U(t)$ then the domain of A is dense in X and A is a closed linear operator.

Proof.

For every $x \in X$ set

$$x_t = \frac{1}{t}\int_0^t U(s)x\ ds.$$

We know that for $x \in X$, $\int_0^t U(s)xds \in D(A)$ and $A\left(\int_0^t U(s)xds\right) = U(t)x - x$. So $x_t \in D(A)$ for every $t > 0$. Also we have For $x \in X$, $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} U(s)x\ ds = U(t)x$. Then $x_t \to x$ as $t \downarrow 0$. Thus $\overline{D(A)}$, the closure of $D(A)$, equals X. The linearity of A is evident. To prove its closedness let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. We have

$$U(t)x_n - x_n = \int_0^t U(s)Ax_n\ ds.$$

We have that $U(s)$ for $s \in [0,t]$ is uniformly bounded for any fixed $t > 0$ which implies

$$U(s)Ax_n \to U(s)y \qquad \text{uniformly for } s \in [0,t].$$

Namely we have

$$\|U(s)Ax_n - U(s)y\| \le \|U(s)\|\,\|Ax_n - y\| \overset{n\to\infty}{\to} 0$$

uniformly in s in a compact set. This implies that the right hand side in $U(t)x_n - x_n = \int_0^t U(s)Ax_n ds$ converges to $\int_0^t U(s)yds$. Also the limit on the left exists and we have

$$U(t)x - x = \int_0^t U(s)y\ ds$$

which implies

$$\frac{U(t)x - x}{t} = \frac{1}{t}\int_0^t U(s)y\ ds.$$

Then the limit on the right exists and goes to y. Thus we see that $x \in D(A)$ and $Ax = y$. ■

1.4.3 Some special class of C_0-semigroup.

Definition 1.4.3.1 **(**Definition 6.2.1,[41]**).** A C_0-semigroup $\{U(t), t \geq 0\}$ is called compact if for all $t > 0$, $U(t)$ is compact operator and eventually compact if there exists $t_0 > 0$, such that $U(t_0)$ is compact.

Theorem 1.4.3.1 (Th 2.3.2, [39]. Let $U(t)$ be a C_0-semigroup. If $U(t)$ is compact for $t > t_0$, then $U(t)$ is continuous in the uniform operator topology for $t > t_0$.

Proof.

Let $\|U(s)\| \leq M$ for $0 \leq s \leq 1$ and let $\varepsilon > 0$ be given. If $t > t_0$ then the set $U_t = \{T(t)x : \|x\| \leq 1\}$ is compact and therefore, there exist $x_1, x_2, ..., x_N$ such that the open balls with radius $\frac{\varepsilon}{2(M+1)}$ centered at $U(t)x_j$, $1 \leq j \leq N$ cover U_t. From the strong continuity of $U(t)$ it is clear that there exists an $0 < h_0 \leq 1$ such that

$$\|U(t+h)x_j - U(t)x_j\| < \frac{\varepsilon}{2} \text{ for } 0 \leq h \leq h_0 \text{ and } 1 \leq j \leq N.$$

Let $x \in X$, $\|x\| \leq 1$, then there is an index j, $1 \leq j \leq N$ (j depending on x) such that

$$\|U(t)x - U(t)x_j\| < \frac{\varepsilon}{2(M+1)}.$$

Thus, for $0 \leq h \leq h_0$ and $\|x\| \leq 1$, we have

$$\begin{aligned} \|U(t+h)x - U(t)x\| &\leq \|U(h)\| \, \|U(t)x - U(t)x_j\| + \|U(t+h)x_j - U(t)x_j\| \\ &\quad + \|U(t)x_j - U(t)x\| \\ &< \varepsilon \end{aligned}$$

which proves the continuity of $U(t)$ in the uniform operator topology for $t > t_0$. ■

1.5 Differential Inclusions.

Differential inclusions serve as models for many dynamical systems. Obviously, any process described by an ordinary differential equation $x'(t) = f(x)$ can be described

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by a differential inclusion with right-hand side $F(x) = \{f(x)\}$. Moreover, differential inclusions play a crucial role in the theory of differential equations with a discontinuous right-hand side. The investigation of such equations is of great importance since they model the performance of various mechanical and electrical devices as well as the behavior of automatic control system.

Differential inclusion takes the form

$$\begin{cases} x'(t) \in F(t, x(t)) \quad a.e. \text{ on } J = [0, b] \subseteq \mathbb{R}, \\ x(0) = x_0. \end{cases} \tag{1.5.1}$$

Where F is a set-valued map from $J \times D$ to the nonempty subsets of a real Banach space X, $D \subseteq X$ is closed and $x_0 \in D$. We have first to agree on what we shall call a solution to such differential inclusion. In the case of differential equations, there is no ambiguity since the derivative $x'(\cdot)$ of a solution $x(\cdot)$ to the differential equation

$$x'(t) = f(t, x(t))$$

inherits the regularity properties of the map f and of the function $x(\cdot)$. This is no longer in the case with differential inclusions and is one of the reasons why their study brings more difficulties than that of the ordinary differential equations. Solutions to differential inclusion (1.5.1) are understood in the Carathéodory sense, i.e., absolutely continuous functions verifying (1.5.1) almost everywhere. For some works done all differential with initial condition or nonlocal conditions we refer to [3,4,7,9,10,32].

Example 1.5.1 ***(Example 2.5.1, [19]).***

Let

$$F : \mathbb{R} \to \mathcal{P}(\mathbb{R}) \backslash \{\phi\}$$

be defined by

$$F(x) = \begin{cases} \{-1\} & \text{if } x > 0 \\ \{-1, 1\} & \text{if } x = 0 \\ \{1\} & \text{if } x < 0 \end{cases}$$

notice that F is $u.s.c.$, with compact values (not convex) and there is no solution to the differential inclusion

$$(*) \begin{cases} x'(t) \in F(x(t)) \quad a.e. \text{ on } J = [0, \infty) \\ x(0) = 0. \end{cases}$$

If we convexity, i.e.

$$F(0) = [-1, 1],$$

then the zero function is the unique solution for $(*)$ which is continuously differentiable. If we change $(*)$ to

$$(**) \begin{cases} x'(t) \in F(x(t)) \quad a.e. \text{ on } J = [0, \infty) \\ x(0) = 1, \end{cases}$$

then the function defined by

$$x(t) = (1 - t)\chi_{[0,1]}(t); t \in [0, \infty)$$

is a solution for $(**)$ which is not continuously differentiable at $t = 1$.

Also, it is important to introduce an other kind of differential inclusions which is differential inclusion with a nonlocal condition that takes the form

$$\begin{cases} x'(t) \in F(t, x(t)) \quad a.e. \text{ on } J = [0, b], \\ x(0) = g(x), \end{cases} \tag{1.5.2}$$

where F is a set–valued map defined from $J \times X$ to the family of all nonempty subsets of a Banach space X and g is a function from the space of all continuous functions from J into the space X. The study of existence of solutions for differential inclusions with non-local conditions was motivated by physical problems (see [6,22]). Moreover, it is a generalization for the classical differential inclusions with local conditions (1.5.1).

1.6 Functional Differential Inclusions.

Differential inclusions express that at every instant the velocity of the system depends upon its state at this instant. Differential inclusions with memory, or, as they are also called, functional differential inclusions, express that the velocity depends not only on the system at this instant, but depends upon the history of the trajectory until this instant. In what follows, we will denote by $C(J, X)$ the space of continuous mappings $u : J \to X$ equipped with the norm of uniform convergence; $\|u\|_{C(J,X)} = \sup_{t \in J} \|u(t)\|$.

To formalize this concept, we give the following definition.

Definition 1.6.1 *(Page 204,[4]). Differential inclusion with memory or 'infinite delay' describes the dependence of the velocity $x'(t)$ upon the history $\tau(t)x$ of $x(\cdot)$ up to time t through a set-valued map F from a subset $\Omega \subseteq \mathbb{R} \times C((-\infty, 0], X)$ to X, where the function $\tau(t)x : C((-\infty, t], X) \to C((-\infty, 0], X)$ is defined by:*

$$(\tau(t)x)(s) = x(t+s), \ \forall s \in (-\infty, 0], \ \forall x \in C((-\infty, t], X).$$

Solving a functional differential inclusion is the problem of finding an absolutely continuous function $x(\cdot) \in C((-\infty, b], X)$ satisfying

$$x'(t) \in F(t, \tau(t)x); \ a.e. \text{ on } J = [0, b].$$

In this regard , Ibrahim and Reem in [3] proved the existence of mild solution of a semilinear functional differential inclusion of a first order with finite delay in the case when the kernel is not necessarily compact given by the formula

$$(J_\Psi) \begin{cases} x'(t) \in A(t)x(t) + F(t, \tau(t)x) & a.e. \text{ on } J, \\ x(t) = \Psi(t) - g(x), & \forall t \in [-r, 0], \end{cases}$$

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where $\Psi \in C_0$. Here $J = [0,b]$, $C_0 = C([-r,0],E)$, $\{A(t) : t \in J\}$ is the family of densely defined, linear operators on the Banach space E, which generates an evolution operator $T : \Delta = \{(t,s) \in J \times J : 0 \leq s \leq t \leq b\} \to \pounds(E)$ (the space of bounded linear operators from E into itself), $F : J \times C_0 \to P_{ck}(E)$ and for any $t \in J$, $\tau(t)$ is the mapping from $C([-r,b],E)$ to C_0 defined by $\tau(t)u(s) = u(s+t)$, for all $s \in [-r,0]$ and $u \in C([-r,b],E)$.

Differential inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, and so forth. Functional differential inclusions with fractional order are first considered by El Sayed and Ibrahim [21]. Very recently Agarwal [1] have considered some class of initial value problems for fractional semilinear functional differential equations and inclusions with both cases of finite and infinite delay.

In [25] Henderson and Ouahab, used the Filippov's Theorem to prove an existence result, for the following initial value problem or (Cauchy problem) of fractional differential inclusion with finite delay in finite dimensional Banach space $\mathbb{R}$,

$$\begin{cases} {}^{L}D_t^q x(t) \in F(t,\tau(t)x) & a.e.\ \forall t \in\ J, \\ x(t) = \Psi(t), & \forall t \in [-r,0], \end{cases} \tag{1.6.1}$$

where $0 < q < 1$ and $\Psi \in C([-r,0],\mathbb{R})$, $\Psi(0) = 0$. Here $J = [0,b]$, $A = 0$,

$$F : J \times C([-r,0],\mathbb{R}) \to \mathcal{P}(\mathbb{R}) \backslash \{\phi\},$$

and ${}^{L}D^q x(t)$ is the Riemann-Liouville fractional derivative of order q to the function x at the point t.

Indeed Henderson and Ouahab [25] proved the following local existence result of (1.6.1).

Theorem 1.6.1 *(Th 5.2 , [25]). Suppose the following hypotheses are satisfied,*

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[A$_1$] $F : J \times C([-r,0],\mathbb{R}) \to \mathcal{P}_k(\mathbb{R})$ is a multifunction such that, for all $x \in C_0$, $t \to F(t,x)$ is measurable.

[A$_2$] There exists a function $p \in L^1(J,\mathbb{R}^+)$ such that, for all $x, y \in C_0$

$$H(F(t,x),F(t,y)) \leq p(t) \left\| x-y \right\|_{C([-r,0],\mathbb{R})}, a.e.\ t \in J.$$

$$d(0,F(t,0)) \leq p(t), \forall a.e.\ t \in J, \ \ |{}_0J_t^q p| < \infty.$$

If

$$\frac{1}{\Gamma(q)} \sup \left\{ |{}_0J_t^q p(t)| : t \in J \right\} < 1,$$

then the Cauchy problem (1.6.1) has at least one solution on $C([-r,b],\mathbb{R})$.

A global existence result of (1.6.1) on unbounded interval $[0,\infty)$ is proved in the following theorem,

Theorem 1.6.2 *(Theorem5.5,[25]). Suppose the following hypotheses are satisfied,*

[A$_1$]$^\infty$ $F : [0,\infty) \times C_0 \to \mathcal{P}_k(E)$ is a multifunction is an upper Carathéodory map, that is for every $x \in C([-r,0],\mathbb{R}), t \to F(t,x)$ is measurable, for almost $t \in [0,\infty)$, $x \to F(t,x)$ is upper semicontinuous.

[A$_2$]$^\infty$ There exists a functions $p, q \in C\left([0,\infty),\mathbb{R}^+\right)$ such that for all $u \in C([-r,0],\mathbb{R})$

$$\left\| F(t,u) \right\|_{\mathcal{P}(E)} = \sup \left\{ |v| : v \in F(t,u) \right\} \leq p(t) + q(t) \left\| u(0) \right\|,$$

for $t \in [0,\infty)$ and $u \in C([-r,0],\mathbb{R})$.

[A$_3$]$^\infty$ For all $R > 0$, there exists $\mathfrak{l}_R \in L^1_{loc}([0,\infty),\mathbb{R}^+)$ such that

$$H(F(t,x),F(t,y)) \leq \mathfrak{l}_R(t) \left\| x-y \right\|_{C([-r,0],\mathbb{R})} \forall x,y \in C([-r,0],\mathbb{R}) \text{ and } \left\| x \right\|, \left\| y \right\| \leq R,$$

and

$$d(0,F(t,0)) \leq \mathfrak{l}_R(t), \forall a.e.\ t \in [0,\infty), |{}_0J_t^q \mathfrak{l}_R| < \infty.$$

If

$$\frac{1}{\Gamma(q)} \sup \{|{}_0J_t^q l_R(t)| : t \in [0,n], n \in \mathbb{N}\} < 1,$$

then the (IVP), (1.6.1) has a global solution on $[-r, \infty)$.

The compactness of the set of solutions of (1.6.1) is established in the following theorem.

Theorem 1.6.3 *(Th 6.3, [25]). Assume that the conditions of Theorem (1.6.1) are satisfied such that the function $p \in C(J, \mathbb{R}^+)$ and $F : J \times C([-r,0], \mathbb{R}) \to \mathcal{P}_{ck}(\mathbb{R})$. Then the solution set of the problem (1.6.1) is compact subset of $C([-r,b], \mathbb{R})$ and*

$$S_J : C([-r,0], \mathbb{R}) \to \mathcal{P}(C([-r,b], \mathbb{R}))$$

is Hausdorff continuous.

In infinite dimensional Banach space more generally than (1.6.1), Agarwal [1] proved the existence of mild solutions of a semilinear functional fractional differential inclusion with finite delay in the case when the kernel is compact given by the formula

$$\begin{cases} {}^L D_t^q x(t) \in A(t)x(t) + F(t, \tau(t)x) & a.e. \forall t \in J, \\ x(t) = \Psi(t), & \forall t \in [-r, 0], \end{cases} \tag{1.6.2}$$

where ${}^L D_t^q$ is the standard Riemann-Liouville fractional derivative. $F : J \times C_0 \to \mathcal{P}(E)$ is a multivalued function. $\mathcal{P}(E)$ is the family of all nonempty subsets of E. $A : D(A) \subset E \to E$ is a densely defined (possibly unbounded) operator generating a strongly continuous semigroup $\{T(t), t \geq 0\}$ of bounded linear operators from E into E. $\Psi : [-r, 0] \to E$ is a given continuous function such that $\Psi(0) = 0$ and $(E, |.|)$ is a real separable Banach space.

In the following theorem the existence of solutions of (1.6.2) with a convex valued right-hand side is proved.

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Theorem 1.6.4 *(Theorem5.2, [1]). Assume the following .*

[H_1] $F : J \times C_0 \to \mathcal{P}_{ck}(E)$ is Carathéodory.

[H_2] The semigroup $\{T(t)\}_{t\in J}$ is compact for $t > 0$.

[H_3] There exists a functions $p, q \in C(J, \mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}(E)} \le p(t) + q(t)\|u\|_C \ , \text{ for } a.e., t \in \ J, \text{ and each } u \in C([-r,0],E).$$

Then the problem (1.6.2) has at least one mild solution on Θ.

1.7 Some Important Facts

In this section we present some important fact that we will need later.

1.7.1. Hölder's Inequality. Let X be a Banach space, (S, σ, μ) be a measurable space, $E \in S$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^q(E, X)$ and $g \in L^p(E, X)$, then $fg \in L^1(E, X)$ and

$$\int_E \|f(t)g(t)\|\, d\mu \le \left(\int_E \|f(t)\|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_E \|g(t)\|^q\, d\mu\right)^{\frac{1}{q}}.$$

1.7.2. Young's Inequality. Let X be a Banach space, $E = [a, b]$, $k \in L^1(E, \mathbb{R})$ and $f \in L^p(E, X)$ for some $p \in [1, \infty)$, then $k * f \in L^p(E, X)$ and

$$\|k * f\|_p \le \|k\|_{L^1(E,\mathbb{R})} \|f\|_{L^p(E,X)},$$

where $k * f$ is the convolution of k and f, which is defined by

$$(k * f)(t) = \int_a^b k(t-s)f(s)ds.$$

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1.7.3. Fatou's Lemma. Let (f_n) be a sequence of non-negative measurable functions on the measurable space (S, σ, μ) and let $f(t) = \lim_{n\to\infty} f_n(t)$ $a.e.$ Then f is measurable and

$$\int_S f(t)d\mu \leq \liminf_{n\to\infty} \int_S f_n(t)d\mu.$$

Corollary 1.7.1 *Let (f_n) be a sequence of real-valued measurable functions on the measurable space (S, σ, μ). If there is an integrable function g such that $f_n \leq g$, a.e. for all $n \geq 1$, then*

$$\limsup_{n\to\infty} \int_S f_n(t)d\mu \leq \int_S \limsup_{n\to\infty} f_n(t)d\mu.$$

Take $z_n = g - f_n$ and apply Fatou's Lemma.

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Chapter 2

Continuous selections of solution sets of semilinear differential inclusions of fractional order.

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2.1 Introduction.

Let $q \in (0,1]$, b be a positive real number, $J = [0,b]$, E be a separable real Banach space, $C(J,E)$ be the Banach space of E-valued continuous functions on J with the uniform norm $\|x\| = \sup\{\|x(t)\|,\ t \in [-r,0]\}$, $F : J \times E \to P_{ck}(E)$ and $A : D(A) \subseteq E \to E$ be the infinitesimal generator of a C_0-semigroup of bounded linear operator $\{T(t) : t \geq 0\}$ in E. Consider the following fractional functional semilinear differential inclusion:

$$(P_\zeta) \quad \begin{cases} D^q x(t) \in Ax(t) + F(t, x(t)), & a.e.\ t \in J, \\ \quad x(0) = \zeta, \end{cases} \tag{2.1.1}$$

where ζ is a given element in E.

It is known that a solution of (2.1.1) is a continuous function $x_\zeta : J \to E$ such that:

$$x_\zeta(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds, \qquad t \in J, \tag{2.1.2}$$

where

$$f \in S_{F(.,\tau(.)x)}\ ,\ S_{F(.,\tau(.)x)} = \{f \in L^1(J,E) : f(t) \in F(t, \tau(t)x), a.e.\ t \in J\},$$
$$K_1(t) = \textstyle\int_0^\infty \xi_q(\theta) T(t^q\theta) d\theta,$$
$$K_2(t) = q \textstyle\int_0^\infty \theta \xi_q(\theta) T(t^q\theta) d\theta,$$
$$\xi_q(\theta) = \tfrac{1}{q} \theta^{-1-\frac{1}{q}} \overline{w_q}\left(\theta^{-\frac{1}{q}}\right) \geq 0,$$
$$\overline{w_q}(\theta) = \tfrac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \tfrac{\Gamma(nq+1)}{n!} \sin(n\pi q), \theta \in (0,\infty), \text{ and } \textstyle\int_0^\infty \xi_q(\theta) d\theta = 1.$$

Let $S_\zeta = \{x_\zeta : x_\zeta \text{ a solution for } (P_\zeta)\}$.

It is known that, under appropriate assumptions that for any $\zeta \in E$, the set S_ζ is not empty. In this chapter we prove that the multivalued function $\zeta \to S_\zeta$ has a continuous selection, that is, there is a continuous function $u : E \to C(J,E)$ such that $u(\zeta) \in S_\zeta$.

We would like to refer that Cernea [15] showed, in finite dimensional spaces, the existence of continuous selections for a fractional differential inclusion (2.1.1) of

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order $q \in (1,2)$ and when $A = 0$. So, our technique allows to extend the result of Cernea [15] to infinite dimensional spaces.

2.2 Preliminaries And Notations.

Let $L^1(J,E)$ be the space of $E-$valued Bochner integrable functions on J with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\| \, dt$, $P_b(E) = \{B \subseteq E : B$ is nonempty and bounded$\}$, $P_{cl}(E) = \{B \subseteq E : B$ is nonempty and closed$\}$, $P_k(E) = \{B \subseteq E : B$ is nonempty and compact$\}$, $P_{cl,cv}(E) = \{B \subseteq E : B$ is nonempty, closed and convex$\}$, $P_{ck}(E) = \{B \subseteq E : B$ is nonempty, convex and compact$\}$, $conv(B)$ (respectively, $\overline{conv}\ (B)$) be the convex hull (respectively, convex closed hull in E) of a subset B.

We need the following Lemmas:

Lemma 2.2.1 *([32], Theorem 8.2.8). Let (Ω, A, μ) be a complete $\sigma-$finite measure space, X a complete separable metric space and $F : \Omega \to 2^X \setminus \{\phi\}$ be a measurable multivalued function with nonempty closed images. Consider a multivalued function G from: $\Omega \times X$ to $P(Y)$, Y is a complete separable metric space such that for every $x \in X$ the multivalued function $w \to G(w,x)$ is measurable and for every $w \in \Omega$ the multivalued function $x \to G(w,x)$ is continuous. Then the multivalued function $w \to \overline{G(w,F(w))}$ is measurable. In particular for every measurable single-valued function $z : \Omega \to X$, the multivalued function $w \to G(w,z(w))$ is measurable and for every Caratheodory single-valued function $\varphi : \Omega \times X \to Y$, the multivalued function $w \to \overline{\varphi(w,F(w))}$ is measurable.*

For more about multivalued functions we can see [4,5,12,20,28,29,34].

Lemma 2.2.2 *([12], Lemma 2.3). Let S be a separable metric space and $D(J,E)$ be the family of all decomposable subsets of $L^1(J,E)$. Let $F^* : J \times S \to P(E)$ be a* *closed valued measurable multifunction such that, for any* $t \in J$, *the multifunction* $s \to F^*(t, s)$ *is l.s.c. Let* $G : S \to D(J, E)$ *be defined by:*

$$G(s) = \left\{ v \in L^1(J, E) : v(t) \in F^*(t, s),\ a.e. \right\}.$$

Then the multifunction G *is l.s.c. with nonempty closed values if and only if there exists a continuous mapping* $p : S \to L^1(J, \mathbb{R}^{\geq 0})$ *such that:*

$$d(0, F^*(t, s)) \leq p(s)(t) \quad a.e.\ on\ J\ and\ for\ all\ s \in S.$$

Lemma 2.2.3 *([13], Theorem III-41). Let* (T, Γ) *be a measurable space,* $G : T \to 2^E$ *be a measurable closed valued multifunction and* $g : T \to E$ *be a measurable function. If the multivalued function*

$$U(t) = \{x \in G(t) : \|g(t) - x\| = d(g(t), G(t))\},$$

has a nonempty values, then it is measurable and hence there is a measurable function $z : T \to E$ *such that* $z(t) \in U(t), a.e.$ *i.e.*

$$\|g(t) - z(t)\| = d(g(t), G(t)),\ a.e.$$

Lemma 2.2.4 *([12], Lemma 2.4). Let* S *be a separable metric space and* $D(J, E)$ *be the family of all decomposable closed subsets of* $L^1(J, E)$. *Let* $G : S \to D(J, E)$ *be l.s.c. multifunction with closed decomposable values and let* $\varphi : S \to L^1(J, E)$, $\eta : S \to L^1(J, \mathbb{R}^+)$ *be continuous such that the multifunction* $H : S \to D(J, E)$ *defined by:*

$$H(s) = cl \left\{ v \in G(s) : \|v(t) - \varphi(t)\| \leq \eta(s)(t),\ a.e. \right\},$$

has nonempty values.

Then H *has a continuous selection, i.e., there exists a continuous mapping* $h : S \to L^1(J, E)$ *such that:*

$$h(s) \in H(s), \qquad \forall s \in S.$$

Lemma 2.2.5 *([18]). Let (X, d) be a complete metric space. If $R : X \to P_{cl}(X)$ is contraction, then R has a fixed point.*

Definition 2.2.1 *([32]). A sequence $\{f_n : n \in \mathbb{N}\} \subseteq L^1(J, E)$ is said to be semi-compact if*

(i) It is integrably bounded, i.e. there is $\beta \in L^1(J, \mathbb{R}^+)$ such that $||f_n(t)|| \leq \beta(t)$, for $a.e.$ $t \in J$ and for every $n \in \mathbb{N}$.

(ii) The set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in E for $a.e.$ $t \in J$.

We recall one fundamental result which follows from Durford-Petties Theorem.

Lemma 2.2.6 *([32, proposition4.2.1]). Every semicompact sequence in $L^1(J, E)$ is weakly compact in $L^1(J, E)$.*

The proof of the following theorem is known., and we will recall it.

Theorem 2.2.1 *Let $F : J \times E \to P_{ck}(E)$ and A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t) : t \geq 0\}$ in E . Assume that the following condition are satisfied*

[HF$_1$] The multivalued function $F : J \times E \to P_{cl}(E)$ has the property that for every $x \in E$, $t \to F(t, x)$ is measurable.

[HF$_2$] There exists $k \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+), \sigma \in (0, q)$ such that for every $x, y \in E$

$$H(F(t, x), F(t, y)) \leq k(t) \parallel x - y \parallel_E a.e., \text{ for } t \in J,$$

and

$$H(\{0\}, F(t, 0)) \leq k(t) \; a.e., \text{ for } t \in J.$$

Then, for any $\zeta \in E$, the problem (P_ζ) has at least one mild solution on $J = [0, b]$ provided that,

$$M \frac{||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}}{\Gamma(q)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}} < 1, \tag{2.2.1}$$

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where M is a positive number such that $\|T(t)\|_{\mathcal{L}(E,E)} \leq M$ and $\eta = \frac{q-\sigma}{1-\sigma}$.

Proof.

At first from Lemma (2.2.1), [HF$_1$] and [HF$_2$] we conclude that for every $x \in C(J,E)$ the multivalued function $t \to F(t,x(t))$ is measurable with closed values, then by Theorem (1.2.2.1), there is a measurable selection for the multifunction $t \to F(t,x(t))$ and the set $S_{F(\cdot,x(.))}$ is nonempty. In order to transform the problem (2.1.1) into a fixed point problem, we consider the multifunction $N_\zeta : C(J,E) \to 2^{C(J,E)} \backslash \{\phi\}$ which is defined as: $y \in N_\zeta(x)$ if and only if

$$y(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds, \; t \in J.$$

We shall show that N_ζ satisfies the assumptions of Lemma (2.2.5). We divide the proof into two steps.

First Step. The values of N_ζ are closed.

Let $x \in C(J,E)$, $\{y_n\}_{n\in\mathbb{N}} \in N_\zeta(x)$ such that $y_n \to y$ in $C(J,E)$. Then for any $n \geq 1$ there exists $f_n \in S_{F(\cdot,x(.))}$ such that

$$y_n(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(s) ds, \; t \in J.$$

Since $F(t,0)$ is closed, from [HF$_2$] we conclude that for any $n \geq 1$ and for a.e. $t \in J$

$$\begin{aligned} ||f_n(t)|| &\leq d(0, f_n(t)) \\ &\leq H(\{0\}, F(t,x(t))) \\ &= H(\{0\}, F(t,0)) + H(F(t,0), F(t,x(t))) \\ &\leq k(t) + k(t)||x(t)|| \\ &\leq k(t)(1 + ||x||_{C(J,E)}). \end{aligned}$$

This show that the set $\{f_n \; : n \geq 1\}$ is integrably bounded. Moreover, since the values of F are compact, then for almost $t \in J$, the set $\{f_n(t) : n \geq 1\}$ is relatively compact in $L^1(J,E)$. Therefore, the set $\{f_n(t) : n \geq 1\}$ is semicompact and hence, by Lemma

(2.2.6), it is weekly relatively compact in $L^1(J, E)$. Then, there exists a subsequence still denoted $\{f_n\}$ which converges weakly to a function $f \in L^1(J, E)$. From Mazur's lemma, for every natural number j there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = k_0(j), ..., j$ such that $\sum_{k=j}^{k_0} \lambda_{j,k} = 1$, and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0} \lambda_{j,k}\ f_{k,j} \geq 1$ converges strongly to f in $L^1(J, E)$ as $j \to \infty$. So, there is a subsequence of (z_n), denoted again by (z_n), such that $z_n \to f$, $a.e.$ Since F takes convex and closed values we obtain for a.e. $t \in J$

$$f(t) \in \bigcap_{j\geq 1} \overline{\{z_k(t) : k \geq j\}} \subseteq \bigcap_{j\geq 1} \overline{co}\{f_k : k \geq j\} \subseteq F(t, x(t))).$$

Note that, for every $t \in J$, $s \in (0, t]$ and every $n \geq 1$

$$||(t-s)^{\alpha-1} K_2(t-s) f_n(s)|| \leq \frac{M}{\Gamma(\alpha)} |t-s|^{\alpha-1}\ k(s)\ (1 + ||x||) \in L^1((0, t], \mathbb{R}^+).$$

Next taking $\overline{y}_n(t) = \sum_{k=n}^{k_0(n)} \lambda_{n,k}\ y_k$. Then

$$\overline{y}_n(t) = K_1(t)\zeta + \int_0^t (t-s)^{\alpha-1} K_2(t-s) z_n(s) ds, t \in J.$$

Observe that for any $t \in J$, $\overline{y}_n(t) \to y(t)$. and $z_n(t) \to f(t), a.e.$ Note that, the continuity of $K_2(t)$ implies that for every $t \in J, K_2(t-s)z_n(s) \to K_2(t-s)f(s)$, for $s \in (0, t)$. Therefore, by passing to the limit as $n \to \infty$ we obtain from the Lebesgue dominated convergence theorem that

$$y(t) = K_1(t)\zeta + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, t \in J.$$

So, $y \in N_\zeta(x)$.

Second step. N_ζ is contraction, that is there exists $1 > \rho > 0$, such that

$$H\ (N_\zeta(x_1), N_\zeta(x_2)) < \rho \left\| x_1 - x_2 \right\|, \forall x_1, x_2 \in C(J, E).$$

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Let $x_1, x_2 \in C(J,E)$ and $y_1 \in N_\zeta(x_1)$. Then there exists $f_1 \in S_{F(\cdot,x_1(.))}$ such that,

$$y_1(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f_1(s)ds, \ \ t \in J.$$

Now let $U : J \to 2^E$ be a multivalued function defined by

$$U(t) = \{x \in F(t, x_2(t)) : d(x, f_1(t)) = d(f_1(t), F(t, x_2(t))\}.$$

Since the values of F are compact, then the values of U are nonempty. By applying Lemma (2.2.3), there is a measurable function $f_2 : J \to E$ such that

$$\|f_1(t) - f_2(t)\| = d(f_1(t), F(t, x_2(t)) \text{ and } f_2(t) \in F(t, x_2(t)), a.e.$$

Note that by [HF$_2$] we get

$$\begin{aligned} \|f_1(t) - f_2(t)\| &\leq H(F(t, x_1(t)), F(t, x_2(t))) \\ &\leq k(t) \|x_1(t) - x_2(t)\|_E, \ t \in J \\ &\leq k(t) \|x_1 - x_2\|_{C(J,E)}, \ \ t \in J. \end{aligned}$$

Let us define

$$y_2(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f_2(s)ds, \ \ t \in J.$$

Then, for any $t \in J$, we have

$$\begin{aligned} ||y_1(t) - y_2(t)|| &\leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} ||f_1(s) - f_2(s)||ds \\ &\leq \frac{M}{\Gamma(q)} \|x_1 - x_2\|_{C(J,E)} \int_0^t (t-s)^{q-1} k(s)ds \\ &\leq \frac{M}{\Gamma(q)} \|x_1 - x_2\|_{C(J,E)} ||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \frac{b^{\ q-\sigma}}{\eta^{1-\sigma}}. \end{aligned}$$

By analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H(N_\zeta(x_1), N_\zeta(x_2)) \leq \frac{M}{\Gamma(q)} \|x_1 - x_2\|_{C(J,E)} ||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \frac{b^{\ q-\sigma}}{\eta^{1-\sigma}}.$$

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Invoking to (2.1.3), the last relation gives us

$$H(N_\zeta(x_1), N_\zeta(x_2)) < \rho \|x_1 - x_2\|_{C(J,E)}, 0 < \rho < 1$$

This proves that N_ζ is contraction, and thus, by Lemma (2.2.5), N_ζ has a fixed point which is a mild solution of problem (R_ζ). ■

2.3 Main Result

In the following theorem, we show that there is a continuous selection for the multivalued function $\zeta \to S_\zeta$, where S_ζ is the set of solutions of (P_ζ).

Theorem 2.3.1 *Let $F : J \times E \to P_{ck}(E)$ and A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t) : t \geq 0\}$ in E. We assume that $[HF_1]$, $[HF_2]$ and the the Relation (2.1.2) hold. Then, the multivalued function $\zeta \to S_\zeta$ has a continuous selection. That is there is a continuous function $u : E \to C(J, E)$ such that $u(\zeta) \in S_\zeta, \forall \zeta \in E$..*

Proof.

We construct two sequences of continuous functions (u_n), (f_n), $n = 0, 1, 2, ...$ such that:

(i)

$$u_n : E \to C(J, E), \quad f_n : E \to L^1(J, E),$$

(ii)

$$u_0(\zeta)(t) = K_1(t)\zeta,$$

(iii)

$$f_n(\zeta)(t) \in F(t, u_n(\zeta)(t)), \quad a.e.,$$

(iv)

$$u_{n+1}(\zeta)(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(\zeta)(s) ds, \quad t \in J,$$

(v)

$$\|f_n(\zeta)(t) - f_{n-1}(\zeta)(t)\| \leq k(t)\beta_n(\zeta), \quad a.e., \ n \geq 1,$$

where $\beta_0(\zeta) = (M+1)\|\zeta\|$

and $\beta_{n+1}(\zeta) = \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_n(\zeta), \ n = 0, 1, 2, ...$

(vi)

$$d(f_n(\zeta)(t), F(t, \tau(t) u_{n+1}(\zeta))) \leq k(t)\beta_{n+1}(\zeta).$$

In order to define $f_0(\zeta)$, let us consider the multifunction:

$$\begin{aligned} F_0^* &: J \times E \to P_{ck}(E), \\ F_0^*(t,\zeta) &= F(t, u_0(\zeta)(t)), \end{aligned}$$

where $u_0(\zeta)$ is given above.

Let us show that, for any fixed $t \in J$, the multivalued function $\zeta \to F_0^*(t,\zeta)$ is continuous. Indeed, let $t \in J$ be a fixed point. By [HF$_2$] we have:

$$\begin{aligned} H(F_0^*(t,\zeta_1), F_0^*(t,\zeta_2)) &= H(F(t, u_0(\zeta_1)(t)), F(t, u_0(\zeta_2)(t))) \\ &\leq k(t)\|u_0(\zeta_1)(t) - u_0(\zeta_2)(t)\| \\ &= k(t) M||\zeta_1 - \zeta_2||. \end{aligned}$$

So, $H(F_0^*(t,\zeta_1), F_0^*(t,\zeta_2))$ tends to zero, when $\zeta_1 \to \zeta_2$ in $C(J,E)$. Hence, for any $t \in J$, the multivalued function $\zeta \to F_0^*(t,\zeta)$ is continuous and therefore $l.s.c.$

Now, we define two multivalued functions:

$$G_0 : E \to 2^{L^1(J,E)} \ \text{ and } \ H_0 : E \to 2^{L^1(J,E)},$$

where

$$\begin{aligned} G_0(\zeta) &= \left\{v \in L^1(J,E) : v(t) \in F_0^*(t,\zeta), \ a.e.\right\}, \\ H_0(\zeta) &= cl\left\{v \in G_0(\zeta) : \|v(t)\| \leq k(t)\beta_0(\zeta), \ a.e.\right\}. \end{aligned}$$

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Our aim is to prove, by using Lemma (2.2.2), that G_0 is $l.s.c.$ At first, we show that the values of G_0 are decomposable, let $\zeta \in E$, $v_1, v_2 \in G_0(\zeta)$ and A be a Lebesgue measurable subset of J. Then for any $t \in J$

$$(v_1\chi_A + v_2\chi_A)(t) = \begin{cases} v_1(t), & \text{if} \quad t \in A, \\ v_2(t), & \text{if} \quad t \in J \setminus A, \end{cases}$$

Then $(v_1\chi_A + v_2\chi_A) \in G_0(\zeta)$ $a.e.$ Hence, the values of G_0 are decomposable.

Note that, from [HF$_2$] for any $\zeta \in E$ we have for $a.e.$ $t \in J$,

$$\begin{aligned} d\left(0, F_0^*(t,\zeta)\right) &\leq H(\{0\}, F(t, u_0(\zeta)(t))) \\ &\leq H\left(\{0\}, F(t,0)\right) + H\left(F(t,0), F\left(t, u_0(\zeta)(t)\right)\right) \\ &\leq k(t) + k(t)\left\|u_0(\zeta)(t)\right\|_E \\ &\leq k(t)[1 + M||\zeta||] \\ &\leq k(t)\beta_0(\zeta) \qquad (2.3.1) \end{aligned}$$

In order to apply Lemma (2.2.2), we define $p_0 : E \to L^1(J, \mathbb{R}^{\geq 0})$ by $p_0(\zeta)(t) = k(t)\beta_0(\zeta)$. We show that p_0 is continuous. Let $\zeta_1, \zeta_2 \in E$. One obtains

$$\begin{aligned} \left\|p_0(\zeta_1) - p_0(\zeta_2)\right\|_{L^1(J,E)} &= \int_0^b \left\|p_0(\zeta_1)(t) - p_0(\zeta_2)(t)\right\| dt \\ &= \int_0^b k(t)\left[|\beta_0(\zeta_1) - \beta_0(\zeta_2)|\right] dt \\ &= M\left\|\zeta_1 - \zeta_2\right\| \int_0^b k(t)dt, \end{aligned}$$

which implies the continuity of p_0.

Then, from Lemma (2.2.2) and equation (2.3.1) G_0 is $l.s.c.$ Moreover, thanks to [HF$_1$] and [HF$_2$], for any $\zeta \in E$ the set $S^1_{F(.,u_0(\zeta)(.))}$ is nonempty . Moreover, if $v_n \to v$ in $L^1(J,E)$, then there is a subsequence (v_{nk}) such that $v_{nk} \to v, a.e.$. Then by the closedness of $F(t, u_0(\zeta)(t))$, we conclude that the values of G_0 is closed.

Next, let us show that the multivalued function H_0 satisfies the assumptions of Lemma (2.2.4). So, we show that $H_0(\zeta)$ is not empty for any $\zeta \in E$. So, let $\zeta \in E$

be a fixed element. Consider the multivalued function Γ_ζ^0 defined by:

$$\Gamma_\zeta^0(t) = \{x \in F_0^*(t,\zeta) : \|x - 0\| = d(0, F_0^*(t,\zeta))\}.$$

By Lemma (2.2.3), Γ_ζ^0 is measurable with nonempty values. Then, Γ_ζ^0 has a measurable selection, i.e. there is a measurable function $v : J \to E$ such that $v(t) \in \Gamma_\zeta^0(t,\zeta)$, for any $t \in J$ and

$$\|v(t)\| = d(0, F_0^*(t,\zeta)),$$

and consequently, by (2.3.1),

$$\|v(t)\| \le k(t)\beta_0(\zeta),\ a.e.$$

This shows that $v \in H_0(\zeta)$.

Therefore, from Lemma (2.2.4), H_0 has a continuous selection $f_0 : E \to L^1(J,E)$ such that:

$$f_0(\zeta) \in H_0(\zeta), \forall \zeta \in E.$$

Let us show that, for any $\zeta \in E$, the set

$$L_0(\zeta) = \{v \in G_0(\zeta) : \|v(t)\| \le k(t)\beta_0(\zeta),\ a.e.\},$$

is closed. Let (v_n) be a sequence in $L_0(\zeta)$ and $v_n \to v$ in $L^1(J,E)$. Since $G_0(\zeta)$ is closed, then $v \in G_0(\zeta)$. Moreover, $v_n \to v$ in measure, hence there a subsequence (v_{n_k}) of (v_n) such that $v_{n_k} \to v$ almost everywhere. Note that

$$\|v_{n_k}(t)\| \le k(t)\beta_0(\zeta),\ a.e.$$

This implies to $\|v(t)\| \le k(t)\beta_0(\zeta)$, $a.e.$ Therefore, $v \in L_0(\zeta)$. This shows that $L_0(\zeta)$ is closed. Then,

$$f_0(\zeta)(t) \in F_0^*(t,\zeta) = F(t, u_0(\zeta)(t)). \tag{2.3.2}$$

and

$$\|f_0(\zeta)(t)\| \le k(t)\beta_0(\zeta), a.e. \tag{2.3.3}$$

Now, let us define $u_1(\zeta): E \to C(J,E)$ as:

$$u_1(\zeta)(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f_0(\zeta)(s) ds, \quad t \in J.$$

In order to define $f_1 : E \to L^1(J,E)$, let

$$\begin{aligned} F_1^* &: J \times E \to P_{ck}(E), \\ F_1^*(t,\zeta) &= F(t, u_1(\zeta)(t)). \end{aligned}$$

Observe that, for any $t \in J$:

$$\begin{aligned} H(F_1^*(t,\zeta_1), F_1^*(t,\zeta_2)) &= H(F(t,u_1(\zeta_1)(t)), F(t,u_1(\zeta_2)(t))) \\ &\leq k(t) \|u_1(\zeta_1)(t) - u_1(\zeta_2)(t)\|_E \\ &\leq k(t)[M\|\zeta_1 - \zeta_2\| \\ &\quad + \int_0^t (t-s)^{q-1} \|K_2(t-s)\| \|f_0(\zeta_1)(s) - f_0(\zeta_2)(s)\| ds] \\ &\leq k(t)[M\|\zeta_1 - \zeta_2\| \\ &\quad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f_0(\zeta_1)(s) - f_0(\zeta_2)(s)\| ds]. \quad (2.3.4) \end{aligned}$$

Hence by the continuity of f_0, $F_1^*(t,\zeta)$ is continuous and consequently $l.s.c.$ for any $t \in J$.

Now, we define two multivalued functions

$$G_1 : E \to 2^{L^1(J,E)} \text{ and } H_1 : E \to 2^{L^1(J,E)}$$

where

$$\begin{aligned} G_1(\zeta) &= \{v \in L^1(J,E) : v(t) \in F_1^*(t,\zeta), \ a.e.\}, \\ H_1(\zeta) &= cl\{v \in G_1(\zeta) : \|v(t) - f_0(\zeta)(t)\| \leq k(t)\beta_1(\zeta), \ a.e.\}, \end{aligned}$$

where $\beta_1(\zeta) = \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_0(\zeta)$.

As above we can show that the values of G_1 are closed and decomposable.

Note that from $[\mathrm{HF}_2]$ and (2.3.3), for any $\zeta \in E$, we get:

$$\begin{aligned} d\left(0, F_1^*(t,\zeta)\right) &\leq H\left(\{0\}, F\left(t, u_1(\zeta)(t)\right)\right) \\ &\leq H\left(\{0\}, F(t,0)\right) + H\left(F(t,0), F\left(t, u_1(\zeta)(t)\right)\right) \\ &\leq k(t) + k(t)\left\|u_1(\zeta)(t)\right\|_E \\ &\leq k(t)[1 + M \ \|\zeta\| \\ &\quad + \int_0^t (t-s)^{q-1}\left\|K_2(t-s)\right\|\left\|f_0(\zeta)(s)\right\| ds] \\ &\leq k(t)[1 + M\|\zeta\| \\ &\quad + \frac{M}{\Gamma(q)}\beta_0(\zeta)\int_0^t (t-s)^{q-1} k(s)ds] \end{aligned} \tag{2.3.5}$$

But, by Holder inequality, we have:

$$\begin{aligned} \int_0^t (t-s)^{q-1} k(s)ds &\leq \left(\int_0^t (\zeta-s)^{\frac{q-1}{1-\sigma}} ds\right)^{1-\sigma} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\ &\leq \frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}. \end{aligned}$$

Thus, the equation (2.3.5) becomes:

$$d\left(0, F_1^*(t,\zeta)\right) \leq k(t)[1 + M\|\zeta\| + \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_0(\zeta)].$$

Let $p_1 : E \to L^1(J, \mathbb{R}^{\geq 0})$ defined by:

$$p_1(\zeta)(t) = k(t)[1 + M\|\zeta\| + \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_0(\zeta)].$$

By the same method used to prove that p_0 is continuous, we can show that p_1 is continuous. Then from Lemma (2.2.2) G_1 is $l.s.c.$ with nonempty closed values.

Now, let $\zeta \in E$ be fixed. Let us show that $H_1(\zeta)$ is not empty. At first, note that from (2.3.2) and Holder's inequality, for any $t \in J$, we get:

$$\begin{aligned} d\left(f_0\left(\zeta\right)(t), F_1^*\left(t,\zeta\right)\right) &\leq H\left(F_0^*\left(t,\zeta\right), F_1^*\left(t,\zeta\right)\right) \\ &= H\left(F\left(t, u_0(\zeta)(t)\right), F\left(t, u_1\left(\zeta\right)(t)\right)\right) \\ &\leq k(t)\left\|u_0(\zeta)(t) - u_1\left(\zeta\right)(t)\right\|_E \\ &\leq k(t)\int_0^t (t-s)^{q-1} K_2(t-s) f_0\left(\zeta\right)(s)ds \\ &\leq k(t)\frac{M}{\Gamma\left(q\right)}\beta_0(\zeta)\int_0^t (t-s)^{q-1} k(s)ds \\ &\leq k(t)\frac{M}{\Gamma\left(q\right)}\beta_0(\zeta)\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\left\|k\right\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\ &= k(t)\beta_1(\zeta) \end{aligned} \tag{2.3.6}$$

where $\beta_1(\zeta) = \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\left\|k\right\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_0(\zeta)$.

Secondly, let Γ_ζ^1 be a multifunction defined by:

$$\Gamma_\zeta^1(t) = \{x \in F_1^*(t,\zeta) : \|x - f_0(\zeta)(t)\| = d(f_0(\zeta)(t), F_1^*(t,\zeta))\}.$$

Then by Lemma (2.2.3), $\Gamma_\zeta^1(t)$ is a measurable and hence there is a measurable function $v : J \to E$ such that $v(t) \in \Gamma_\zeta^1(t)$, $\forall t \in J$. This implies to

$$\|v(t) - f_0(\zeta)(t)\| = d(f_0(\zeta)(t), F_1^*(t,\zeta)).$$

This equation with (2.3.6) give us:

$$\|v(t) - f_0(\zeta)(t)\| \leq k(t)\beta_1(\zeta), a.e.$$

and $v \in G_1(\zeta)$. Thus $v \in H_1(\zeta)$.

From Lemma (2.2.4), H_1 has a continuous selection $f_1 : E \to L^1(J,E)$ such that:

$$f_1(\zeta) \in H_1(\zeta), \forall \zeta \in E.$$

By arguing as above, we can show that the closedness of $G_1(\zeta)$ implies that the set

$$L_1(\zeta) = \{v \in G_1(\zeta) : \|v(t) - f_0(\zeta)(t)\| \leq k(t)\beta_1(\zeta), \quad a.e.\}.$$

is closed. Hence,

$$f_1(\zeta)(t) \in F_1^*(t,\zeta) = F(t, u_1(\zeta)(t)),$$

and

$$\|f_1(\zeta)(t) - f_0(\zeta)(t)\| \leq k(t)\beta_1(\zeta), a.e. \tag{2.3.7}$$

Suppose that we have constructed $u_0, u_1, ..., u_n, ..., f_0, f_1, ..., f_n, ...$ satisfying (i)→(vi).

Let us define $u_{n+1} : E \to C(J,E)$ as:

$$u_{n+1}(\zeta)(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(\zeta)(s) ds, \quad t \in J,$$

and hence define:

$$F_{n+1}^* : J \times E \to P_{ck}(E), \; by :$$

$$F_{n+1}^*(t,\zeta) = F(t, u_{n+1}(\zeta)(t)),$$

$$G_{n+1} : E \to 2^{L^1(J,E)}, \; by :$$

$$G_{n+1}(\zeta) = \left\{v \in L^1(J,E) : v(t) \in F_{n+1}^*(t,\zeta), \quad a.e.\right\},$$

and

$$H_{n+1} : E \to 2^{L^1(J,E)}, \; by :$$

$$H_{n+1}(\zeta) = cl\left\{v \in G_{n+1}(\zeta) : \|v(t) - f_n(\zeta)(t)\| \leq k(t)\beta_{n+1}(\zeta), \quad a.e.\right\},$$

where $\beta_{n+1}(\zeta) = \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_n(\zeta), n \geq 0.$

Let us show that for any $t \in J$, $t \to F_{n+1}^*(t,\zeta)$ is continuous. So, let $\zeta_1, \zeta_2 \in E$. By $[HF_2]$, we have:

$$\begin{aligned} H\left(F_{n+1}^*(t,\zeta_1), F_{n+1}^*(t,\zeta_2)\right) &= H\left(F(t, u_{n+1}(\zeta_1)(t)), F(t, u_{n+1}(\zeta_2)(t))\right) \\ &\leq k(t)\|u_{n+1}(\zeta_1)(t) - u_{n+1}(\zeta_2)(t)\|_E \\ &\leq k(t)\int_0^t (t-s)^{q-1}\|K_2(t-s)\|\|f_n(\zeta_1)(s) - f_n(\zeta_2)(s)\| ds \\ &\leq k(t)[M\|\zeta_1 - \zeta_2\| \\ &\quad + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f_n(\zeta_1)(s) - f_n(\zeta_2)(s)\| ds]. \end{aligned}$$

Hence by the continuity of f_n, we infer that the multivalued function $t \to F^*_{n+1}(t,\zeta)$ is continuous and therefore $l.s.c.$

Moreover, from $[\mathrm{HF}_2]$ for any $\zeta \in E$ we get:

$$\begin{aligned} d\left(0, F^*_{n+1}(t,\zeta)\right) &\leq H\left(\{0\}, F\left(t, u_{n+1}(\zeta)(t)\right)\right) \\ &\leq H\left(\{0\}, F(t,0)\right) + H\left(F(t,0), F\left(t, u_{n+1}(\zeta)(t)\right)\right) \\ &\leq k(t) + k(t)\left\|u_{n+1}(\zeta)(t)\right\|_E \\ &\leq k(t)[1 + M\|\zeta\| \\ &\quad + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|f_n(\zeta)(s)\right\| ds]. \qquad (2.3.8) \end{aligned}$$

Furthermore, from property (v), for any $t \in J$, one obtains:

$$\begin{aligned} \left\|f_n(\zeta)(t)\right\| &\leq \left\|f_n(\zeta)(t) - f_{n-1}(\zeta)(t)\right\| + \left\|f_{n-1}(\zeta)(t) - f_{n-2}(\zeta)(t)\right\| + ... \\ &\quad + \left\|f_1(\zeta)(t) - f_0(\zeta)(t)\right\| + \|f_0(\zeta)(t)\| \\ &\leq k(t)\beta_n(\zeta) + k(t)\beta_{n-1}(\zeta) + ... + k(t)\beta_1(\zeta) + k(t)\beta_0(\zeta). \end{aligned}$$

Thus, the equation (2.3.8) becomes:

$$d\left(0, F^*_{n+1}(t,\zeta)\right) \leq k(t)[1 + M\|\zeta\| + \frac{M}{\Gamma(q)}\sum_{m=0}^{n}\beta_m(\zeta)\int_0^t (t-s)^{q-1}k(s)ds]. \qquad (2.3.9)$$

Note that, by Holder's inequality, we have:

$$\begin{aligned} \int_0^\zeta (\zeta - s)^{q-1}k(s)ds &\leq \left(\int_0^\zeta (\zeta-s)^{\frac{q-1}{1-\sigma}} ds\right)^{1-\sigma} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\ &\leq \frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}. \end{aligned}$$

Then, by (2.3.9):

$$\begin{aligned} d\left(0, F^*_{n+1}(t,\zeta)\right) &\leq k(t)[1 + M\|\zeta\| + \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\left(\sum_{m=0}^{n}\beta_m(\zeta)\right)] \\ &\leq p_{n+1}(\zeta)(t), \qquad (2.3.10) \end{aligned}$$

where $p_{n+1} : E \to L^1(J, \mathbb{R}^{\geq 0})$ is defined by:

$$p_{n+1}(\zeta)(t) = k(t)[1 + M\|\zeta\| + \frac{M}{\Gamma(q)}\left(\sum_{m=0}^{n}\beta_m(\zeta)\right)\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}].$$

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Observe that for $\zeta_1, \zeta_2 \in E$ we have:

$$\begin{aligned}\|p_{n+1}(\zeta_1) - p_{n+1}(\zeta_2)\|_{L^1(J,E)} &= \int_0^b \|p_{n+1}(\zeta_1)(t) - p_{n+1}(\zeta_2)(t)\| \, dt \\ &= \int_0^b k(t)\{1 + M| \, \|\zeta_1\| - \|\zeta_2\| \, | \\ &\quad + \frac{M}{\Gamma(q)} \left| \left(\sum_{m=0}^{n} \beta_m(\zeta_1) \right) - \left(\sum_{m=0}^{n} \beta_m(\zeta_2) \right) \right| \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \} dt,\end{aligned}$$

This shows that p_{n+1} is continuous.

Then from Lemma (2.2.2) and (2.3.10) G_{n+1} is $l.s.c.$ with nonempty closed values. Moreover, we can show as above that the values of G_{n+1} are decomposable.

In order to apply Lemma (2.2.4) we prove that the values of H_{n+1} are not empty. So, let $\zeta \in E$ be fixed.

For any $t \in J$, by [HF$_2$], one obtains:

$$\begin{aligned} d\left(f_n(\zeta)(t), F^*_{n+1}(t,\zeta)\right) &\leq H\left(F^*_n(t,\zeta), F^*_{n+1}(t,\zeta)\right) \\ &= H\left(F(t, u_n(\zeta)(t)), F(t, u_{n+1}(\zeta)(t))\right) \\ &\leq k(t) \|u_n(\zeta)(t) - u_{n+1}(\zeta)(t)\|_E \\ &\leq k(t) \int_0^t (t-s)^{q-1} \|K_2(t-s)\| \, \|f_n(\zeta)(s) - f_{n-1}(\zeta)(s)\| \, ds] \\ &\leq k(t) \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f_n(\zeta)(s) - f_{n-1}(\zeta)(s)\| \, ds] \\ &\leq k(t) \frac{M}{\Gamma(q)} \beta_n(\zeta) \int_0^t (t-s)^{q-1} k(s) ds \\ &\leq k(t) \frac{M}{\Gamma(q)} \beta_n(\zeta) \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\ &= k(t) \beta_{n+1}(\zeta), \qquad (2.3.11) \end{aligned}$$

where $\beta_{n+1}(\zeta) = \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_n(\zeta)$.

Now, let Γ^{n+1}_ζ be a multivalued function defined on J by:

$$\Gamma^{n+1}_\zeta(t) = \{x \in F^*_{n+1}(t,\zeta) : \|x - f_n(\zeta)(t)\| = d(f_n(\zeta)(t), F^*_{n+1}(t,\zeta))\}.$$

Since the values of F^*_{n+1} are compact, then by Lemma (2.2.3) there is a measurable function $v : J \to E$ such that $v(t) \in \Gamma^{n+1}_\zeta(t), \forall t \in J$.

Therefore, by (2.3.11)

$$\begin{aligned} \|v(t) - f_n(\zeta)(t)\| &= d(f_n(\zeta), F^*_{n+1}(t,\zeta)) \\ &\leq k(t)\beta_{n+1}(\zeta), \ a.e. \end{aligned}$$

Then $v \in G_{n+1}(\zeta)$ and $\|v(t) - f_n(\zeta)(t)\| \leq k(t)\beta_{n+1}(\zeta)$, $a.e.$ Consequently $v \in H_{n+1}(\zeta)$.

From Lemma (2.2.4), H_{n+1} has a continuous selection $f_{n+1} : E \to L^1(J, E)$ such that $f_{n+1}(\zeta) \in H_{n+1}(\zeta), \forall \zeta \in E$.

By arguing as above, we can show that the closedness of $G_{n+1}(\zeta)$ implies that the set

$$L_{n+1}(\zeta) = \{v \in G_{n+1}(\zeta), \|v(t) - f_n(\zeta)(t)\| \leq k(t)\beta_n(\zeta)\}$$

is closed. Then

$$f_{n+1}(\zeta)(t) \in F^*_{n+1}(t,\zeta) = F(t, u_{n+1}(\zeta)(t)), \tag{2.3.12}$$

and

$$\|f_{n+1}(\zeta)(t) - f_n(\zeta)(t)\| \leq k(t)\beta_{n+1}(\zeta). \tag{2.3.13}$$

Therefore, the functions $u_0, u_1, ...u_n, ..., f_0, f_1, ..., f_n$, ...are constructed and satisfying the properties (i)→(vi).

Now, from the property (v), for all $t \in J$ and for all $\zeta \in E$ we have:

$$\begin{aligned}\left\|f_{n+1}(\zeta)(t)-f_n(\zeta)(t)\right\| &\le k(t)\beta_{n+1}(\zeta)\\ &= k(t)\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_n(\zeta)\\ &= k(t)\left[\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\right]^2\beta_{n-1}(\zeta)\\ &.\\ &.\\ &.\\ &= k(t)\left[\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\right]^{n+1}\beta_0(\zeta)\\ &= k(t)\left[\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\right]^{n+1}(M+1)\|\zeta\|. \quad (2.3.14)\end{aligned}$$

Then

$$\left\|f_{n+1}(\zeta)-f_n(\zeta)\right\|_{L^1(J,E)} \le \|k\|_{L^1(J,\mathbb{R}^+)}(M+1)\|\zeta\|\,\delta^{n+1}, \quad (2.3.15)$$

where

$$\delta = \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.$$

Since $0<\delta<1$, then (2.3.15) implies that for any $\zeta \in E$, any two natural number n, m with $n<m$ and any $t \in J$

$$\begin{aligned}\|f_m(\zeta)-f_n(\zeta)\|_{L^1(J,E)} &\le \|f_m(\zeta)-f_{m-1}(\zeta)\|_{L^1(J,E)}+\ldots..+\|f_{n+1}(\zeta)-f_n(\zeta)\|_{L^1(J,E)}\\ &\le ||k||_{L^1(J,\mathbb{R}^+)}[\delta^m+\delta^{m-1}+\ldots..+\delta^{n+1}](M+1)\|\zeta\|\\ &\le ||k||_{L^1(J,\mathbb{R}^+)}\delta^{n+1}[1+\delta+\delta^2+\ldots..+\delta^{m-(n+1)}](M+1)\|\zeta\|\\ &\le ||k||_{L^1(J,\mathbb{R}^+)}\delta^{n+1}\sum_{k=0}^{\infty}\delta^k(M+1)\|\zeta\|\\ &= ||k||_{L^1(J,\mathbb{R}^+)}\delta^{n+1}\frac{1}{1-\delta}(M+1)\|\zeta\|. \quad (2.3.16)\end{aligned}$$

Since $0<\delta<1$, then:

$$\lim_{m,n\to\infty}\|f_m(\zeta)-f_n(\zeta)\|_{L^1(J,E)}=0,$$

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This implies that for any $\zeta \in E$, the sequence $(f_n(\zeta))$ is Cauchy in $L^1(J,E)$. So, there exists a function $f : E \to L^1(J,E)$ such that:

$$\lim_{n\to\infty} f_n(\zeta) = f(\zeta), \quad \forall \zeta \in E.$$

To prove that $f : E \to L^1(J,E)$ is continuous, let $\zeta_1,\ \zeta_2 \in E$ and $\varepsilon > 0$, since $f_n(\zeta_1) \to f(\zeta_1)$ and $f_n(\zeta_2) \to f(\zeta_2)$, there is a natural number $N = N(\zeta_1, \zeta_2)$ such that for $n \geq N$ we have:

$$\|f_n(\zeta_1) - f(\zeta_1)\|_{L^1(J,E)} \leq \frac{\varepsilon}{3}, \tag{2.3.17}$$

and

$$\|f_n(\zeta_2) - f(\zeta_2)\|_{L^1(J,E)} \leq \frac{\varepsilon}{3}. \tag{2.3.18}$$

By the continuity of f_N, there is $\delta > 0$ such that:

$$\|\zeta_1 - \zeta_2\| < \delta \Rightarrow \|f_N(\zeta_1) - f_N(\zeta_2)\|_{L^1(J,E)} < \frac{\varepsilon}{3}. \tag{2.3.19}$$

Then from (2.3.17), (2.3.18), (2.3.19) we have:

$$\begin{aligned}\|f(\zeta_1) - f(\zeta_2)\|_{L^1(J,E)} &\leq \|f(\zeta_1) - f_N(\zeta_1)\|_{L^1(J,E)} + \|f_N(\zeta_1) - f_N(\zeta_2)\|_{L^1(J,E)} \\ &\quad + \|f_N(\zeta_2) - f(\zeta_2)\|_{L^1(J,E)} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

This means that $f : E \to L^1(J,E)$ is continuous.

Now, let $\zeta \in E$ be fixed. From the definition of $u_n(\zeta), u_{n+1}(\zeta)$ and (2.3.15) we get:

$$\begin{aligned}\|u_{n+1}(\zeta) - u_n(\zeta)\|_E &\leq \int_0^t (t-s)^{q-1} \|K_2(t-s)\| \|f_n(\zeta)(s) - f_{n-1}(\zeta)(s)\| ds \\ &\leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f_n(\zeta)(s) - f_{n-1}(\zeta)(s)\| ds \\ &\leq \frac{M}{\Gamma(q)} \delta^n (M+1) \|\zeta\| \int_0^t (t-s)^{q-1} k(s) ds \\ &\leq \frac{M}{\Gamma(q)} \delta^n (M+1) \|\zeta\| \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}. \end{aligned} \tag{2.3.20}$$

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By arguing as in (2.3.16) we can show that $u_n(\zeta)$ is a Cauchy sequence in $C(J,E)$. Hence there is $u(\zeta)\in C(J,E)$ such that $u_n(\zeta)$ converges to $u(\zeta)$ in $C(J,E)$.

Let us define $u: E\to C(J,E)$ such that

$$u(\zeta)=\lim_{n\to\infty}u_n(\zeta).$$

It follows from the fact that $u_n(\zeta)$ converges uniformly to $u(\zeta)$ in $C(J,E)$ that $u(\zeta)$ is continuous.

Next, we prove that

$$f\left(\zeta\right)\left(t\right)\in F\left(t,u\left(\zeta\right)\left(t\right)\right),a.e.$$

So, let $\zeta\in E$. We have, by [HF_2], for $a.e.$:

$$\begin{aligned} d\left(f_n\left(\zeta\right)\left(t\right),F\left(t,u\left(\zeta\right)\left(t\right)\right)\right) &\le H\left(F\left(t,u_n\left(\zeta\right)\left(t\right)\right),F\left(t,u\left(\zeta\right)\left(t\right)\right)\right)\\ &\le k(t)\left\|u_n\left(\zeta\right)\left(t\right)-u\left(\zeta\right)\left(t\right)\right\|_E\\ &\le k(t)\left\|u_n(\zeta)-u(\zeta)\right\|_{C(J,E)}. \end{aligned} \tag{2.3.21}$$

Because $f_n(\zeta)$ converges to $f(\zeta)$ in $L^1(J,E)$, then $f_n(\zeta)$ converges in measure to $f(\zeta)$ and hence we can find a subsequence $f_{n_k}(\zeta)$ of $f_n(\zeta)$ such that

$$f_{n_k}(\zeta)\to f(\zeta),\ a.e.$$

Since $\left\|u_n(\zeta)-u(\zeta)\right\|_{C(J,E)}\to 0$, as $n\to\infty$, the last inequality with (2.2.21) gives us:

$$f\left(\zeta\right)\left(t\right)\in F\left(t,u\left(\zeta\right)\left(t\right)\right),\ a.e. \tag{2.3.22}$$

Now, let $v: E\to C(J,E)$ defined by:

$$v\left(\zeta\right)\left(t\right)=K_1(t)\zeta+\int_0^t (t-s)^{q-1}K_2(t-s)f\left(\zeta\right)(s)ds,\qquad t\in J.$$

Let us show that

$$u(\zeta)(t)=v(\zeta)(t),\ \forall\zeta\in E \text{ and } t\in J.$$

Let $\zeta \in E$. Note that for almost $t \in J$,

$$\begin{aligned} \|f_n(\zeta)(t)\| &\leq H(\{0\}, F(t, u_n(\zeta)(t)) \\ &\leq H(\{0\}, F(t,0)) + H(F(t,0), F(t, u_n(\zeta)(t))) \\ &\leq k(t) + k(t) \|u_n(\zeta)(t)\|_E \\ &\leq k(t) + k(t) \|u_n(\zeta)\|_{C(J,E)}. \end{aligned}$$

Since $u_n(\zeta)$ converges uniformly to $u(\zeta)$ in $C(J, E)$, then $u_n(\zeta)$ is uniformly bounded, hence we can find an integrable function $z_\zeta : J \to [0, \infty)$ such that

$$\|f_n(\zeta)(t)\| \leq z_\zeta(t), \ a.e.$$

Moreover, as above, there is a subsequence $(f_{n_n}(\zeta))$ of $f_n(\zeta)$ such that

$$f_{n_k}(\zeta) \to f(\zeta), \ \ a.e.$$

Then, by the Lebesgue dominated convergence theorem we get for $t \in J$,

$$\lim_{n_k \to \infty} u_{n_k}(\zeta)(t) = v(\zeta)(t).$$

Then

$$v(\zeta)(t) = u(\zeta)(t), \ \ \forall t \in J \text{ and } \zeta \in E.$$

Thus:

$$u(\zeta)(t) = K_1(t)\zeta + \int_0^t (t-s)^{q-1} K_2(t-s) f(\zeta)(s) ds, \qquad t \in J,$$

and

$$u(\zeta) \in S(\zeta), \forall \zeta \in E.$$

This means that $u : E \to C(J, E)$ is a continuous selection for S_ζ and this complete the proof. ■

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Chapter 3

Continuous selections of solution sets of semilinear functional differential inclusions of fractional order.

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3.1 Introduction.

Let $q \in (0,1]$, r, b be two positive real numbers, $J=[0,b]$, $C_r = C([-r,0],\mathbb{R})$ be the Banach space of E-valued continuous functions on $[-r,0]$ with the uniform norm $\|x\| = \sup\{\|x(t)\|,\ t\in[-r,0]\}, C_b = C([-r,b],\mathbb{R})$, $F : J\times C_r \to P_{ck}(\mathbb{R})$ and $A : D(A) \subseteq \mathbb{R} \to \mathbb{R}$be the infinitesimal generator of a C_0-semigroup of bounded linear operator $\{T(t) : t \geq 0\}$ in $\mathbb{R}$. Consider the following fractional functional semilinear differential inclusion:

$$(P_\psi) \quad \begin{cases} D^q x(t) \in Ax(t) + F(t,\tau(t)x)), & a.e.\ t\in\ J, \\ x(t) = \psi(t), & t\in[-r,0], \end{cases} \tag{3.1.1}$$

where D^q is the Caputo derivative of order q for the function x at the point t and ψ is a given element in C_r.

It is known that a solution of (3.1.1) is a continuous function $x_\psi : [-r,b] \to \mathbb{R}$ such that:

$$x_\psi(t) = \begin{cases} \psi(t), & t\in[-r,0], \\ K_1(t)\psi(0) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s)ds, & t\in J, \end{cases} \tag{3.1.2}$$

where

$$f \in S_{F(.,\tau(.)x)}\ ,\ S_{F(.,\tau(.)x)} = \{f\in L^1(J,E) : f(t)\in F(t,\tau(t)x), a.e.\ t\in J\},$$
$$K_1(t) = \int_0^\infty \xi_q(\theta) T(t^q\theta)d\theta,$$
$$K_2(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta,$$
$$\xi_q(\theta) = \tfrac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w_q}\left(\theta^{-\frac{1}{q}}\right) \geq 0,$$
$$\overline{w_q}(\theta) = \tfrac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\tfrac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta\in(0,\infty), \text{ and } \int_0^\infty \xi_q(\theta)d\theta = 1.$$

Let $S_\psi = \{x_\psi : x_\psi \text{ a solution for } (P_\psi)\}$.

Ibrahim and Almoulhim [30] proved, in finite dimensional spaces and under appropriate conditions, that the set S_ψ is not empty.

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In this chapter, we generalize this result to infinite dimensional spaces. Moreover, we show, that the multivalued function $\psi \to S_\psi$ has a continuous selection, that is, there is a continuous function $u : \mathbb{R} \to C_b$ such that $u(\psi) \in S_\psi$.

We would like to refer that in [22], Elshahry proved that the existence of continuous selection for the set of solutions for the functional semilinear differential inclusion.

Also we would like to refer that Cernea [15] showed, in finite dimensional spaces, the existence of continuous selections for a fractional differential inclusion $(P_\zeta\ )$ of order $q \in (1,2)$ and when $A = 0$. So, our obtained results extend the work done by Cerna [15] to the case when there is a delay.

3.2 Preliminaries And Notations.

Let E be a separable real Banach space, $L^1(J,E)$ be the Banach space of (equivalence classes) $E-$valued Bochner integrable functions on J with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\|\,dt$, $P_b(E) = \{B \subseteq E : B$ is nonempty and bounded$\}$, $P_{cl}(E) = \{B \subseteq E : B$ is nonempty and closed$\}$, $P_k(E) = \{B \subseteq E : B$ is nonempty and compact$\}$, $P_{cl,cv}(E) = \{B \subseteq E : B$ is nonempty, closed and convex$\}$, $P_{ck}(E) = \{B \subseteq E : B$ is nonempty, convex and compact$\}$, $conv(B)$(respectively, $\overline{conv}\ (B)$) be the convex hull (respectively, convex closed hull in E) of a subset B.

Ibrahim and Almoulhim [30] proved , in finite dimensional spaces, the existence of solutions for (P_ψ) . Indeed, they proved the following theorem:

Theorem 3.2.1 *Let $E = \mathbb{R}^n$ and H be the Hausdorff distance on $P_{cl}(E)$. If the following conditions hold*

[HA] $A : D(A) \subseteq E \to \mathbb{R}$ is the infinitesimal generator of a C_0-semigroup of bounded linear operator $\{T(t) : t \geq 0\}$ in E.

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[HF$_3$] The multivalued function $F : J \times C_r \to P_{cl}(E)$ has the property that for every $x \in C_r$, $t \to F(t,x)$ is measurable.

[HF$_4$] There exists $k \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, $\sigma \in (0,q)$ such that for every $\varphi, \psi \in C_r$

$$H(F(t,\varphi), F(t,\psi)) \leq k(t) \parallel \varphi - \psi \parallel_{C_r} \ a.e., \ \text{for } t \in J,$$

and

$$d\ (0, F(t,0)) \leq k(t) \ \ a.e., \ \text{for } t \in J.$$

[H$_g$] $g : C_b \to E$ is a function such that there is a positive constant ς with

$$||\ g(z) - g(w)|| \ \leq \varsigma\ ||z - w\ ||_{C_b}\ ,\ \forall z, w \in C_b.$$

Then, the problem:

$$(R_\psi\ ) \quad \begin{cases} {}^cD^q x(t) \in Ax(t) + F(t, \tau(t)x), & a.e.,\ t \in\ J, \\ x(t) = \psi\,(t) - g(x), & \forall t \in [-r, 0]\,, \end{cases}$$

has at least one mild solution on $[-r, b]$ provided that,

$$M\ (\varsigma + \frac{||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}}{\Gamma(q)} \frac{b^{\ q-\sigma}}{\eta^{1-\sigma}}) < 1,$$

where $\|T(t)\|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \leq M$ for some $M > 0$ and $\eta = \frac{q-\sigma}{1-\sigma}$.

3.3 Main Results.

Our first aim is to extend Theorem (3.2.1) to infinite dimensional real separable Banach spaces.

Theorem 3.3.1 *Let $F : J \times C_r \to P_{ck}(E)$ and A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t) : t \geq 0\}$ in E. Assume that the conditions [HF$_3$], [H$_g$] and the following condition are satisfied:*

[HF$_5$] There exists $k \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+), \sigma \in (0, q)$ such that for every $\varphi_1, \varphi_2 \in C_r$

$$H(F(t, \varphi_1), F(t, \varphi_2)) \leq k(t) \parallel \varphi_2 - \varphi_2 \parallel_{C_r} a.e., \text{ for } t \in J,$$

and

$$H(\{0\}, F(t, 0)) \leq k(t) \; a.e., \text{ for } t \in J.$$

Then, for any $\psi \in C_r$, the problem (R_ψ) has at least one mild solution on $[-r, b]$ provided that,

$$M\ (\varsigma + \frac{||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}+)}}{\Gamma(q)} \frac{b^{\ q-\sigma}}{\eta^{1-\sigma}}) < 1, \qquad (3.3.1)$$

where M is a positive number such that $\|T(t)\|_{\mathcal{L}(E,E)} \leq M$ and $\eta = \frac{q-\sigma}{1-\sigma}$.

Proof.

At first from Lemma (2.2.1), [HF$_3$] and [HF$_5$] we conclude that for every $x \in C_b$ the multivalued function $t \to F(t, \tau(t)x)$ is measurable with closed values, then by Theorem (1.2.2.1), it has a measurable selection, and hence the set $S_{F(\cdot,x(.))}$ is nonempty. In order to transform the problem (3.1.1) into a fixed point problem, we consider the multifunction $N_\zeta : C_b \to 2^{C_b} \backslash \{\phi\}$ which is defined as: $y \in N_\psi(x)$ if and only if

$$y(t) = \begin{cases} \psi(t) - g(x), \ t \in [-r, 0], \\ K_1(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds, \ t \in J. \end{cases}$$

We shall show that N_ψ satisfies the assumptions of Lemma (2.2.5). We divide the proof into two steps.

First Step. The values of N_ψ are closed.

Let $x \in C_b$, $\{y_n\}_{n \in \mathbb{N}} \in N_\psi(x)$ such that $y_n \to y$ in C_b .Then, for any $n \geq 1$ there exists $f_n \in S_{F(\cdot,\tau(\cdot)x)}$ such that

$$y_n(t) = \begin{cases} \psi(t) - g(x), \ t \in [-r, 0], \\ K_1(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(s) ds, \ t \in J. \end{cases}$$

Obviously, $y(t) = \psi(t) - g(x);\ t \in [-r, 0]$. Moreover, since $F(t, 0)$ is closed, from [HF_5] we conclude that for any $n \geq 1$ and for a.e. $t \in J$

$$\begin{aligned} |f_n(t)| &= d(0, f_n(t)) \\ &\leq H(\{0\}, F(t, x(t))) \\ &= H(\{0\}, F(t,0)) + H(F(t,0), F(t, x(t))) \\ &\leq k(t) + k(t)||x(t)|| \\ &\leq k(t)(1 + ||x||_{C_b}). \end{aligned}$$

This show that the set $\{f_n\ : n \geq 1\}$ is integrably bounded. Moreover, since the values of F are compact, then for almost $t \in J$, the set $\{f_n(t) : n \geq 1\}$ is relatively compact in $L^1(J, E)$. Therefore the set $\{f_n(t) : n \geq 1\}$ is semicompact and hence it is weekly relatively compact in $L^1(J, E)$. Then, there exists a subsequence, still denoted $\{f_n\}$, which converges weakly to a function $f \in L^1(J, E)$. From Mazur's lemma, for every natural number j there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = k_0(j), \ldots.., j$ such that $\sum_{k=j}^{k_0} \lambda_{j,k} = 1$, and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0} \lambda_{j,k}\ f_{k,j} \geq 1$ converges strongly to f in $L^1(J, E)$ as $j \to \infty$. Then there is a subsequence of z_n, denoted again by z_n, such that $z_n \to f$, $a.e.$ Since F takes convex and closed values we obtain for a.e. $t \in\ J$

$$f(t) \in \bigcap_{j\geq 1} \overline{\{z_k(t) : k \geq qj\}}\ \subseteq \bigcap_{j\geq 1} \overline{co}\{f_k : k \geq j\} \subseteq F(t, x(t))).$$

Note that, for every $t \in J$, $s \in\ (0, t]$ and every $n \geq 1$

$$||(t-s)^{\alpha-1} K_2(t-s) f_n(s)||\ \ \leq\ \frac{M}{\Gamma(\alpha)}\ |t-s|^{\alpha-1}\ k(s)\ (1 + ||x||) \in L^1((0, t], \mathbb{R}^+).$$

Next taking $\overline{y}_n(t) = \sum_{k=n}^{k_0(n)} \lambda_{n,k}\ y_k$. Then

$$\overline{y}_n(t) = \begin{cases} \psi(t) - g(x),\ t \in [-r, 0], \\ K_1(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) z_n(s) ds, t \in J. \end{cases}$$

Observe that for any $t \in J$, $\overline{y}_n(t) \to y(t)$ and $z_n(t) \to f(t)$, $a.e.$ Then, by the continuity of $K_2(t)$ for every $t \in J, K_2(t-s)z_n(s) \to K_2(t-s)f(s)$, for $s \in (0,t)$. Therefore, by passing to the limit as $n \to \infty$ we obtain from the Lebesgue dominated convergence theorem that

$$y(t) = \begin{cases} \psi(t) - g(x), \ t \in [-r, 0], \\ K_1(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, t \in J. \end{cases}$$

So, $y \in N_\psi(x)$.

Second step. N_ψ is contraction, that is there exists $1 > \rho > 0$, such that

$$H \ (N_\psi(x_1), N_\psi(x_2)) < \rho \left\| x_1 - x_2 \right\|, \forall x_1, x_2 \in C_b.$$

Let $x_1, x_2 \in C_r$ and $y_1 \in N_\psi(x_1)$. Then there exists $f_1 \in S_{F(\cdot,\tau(\cdot)x_1)}$ such that,

$$y_1(t) = \begin{cases} \psi(t) - g(x_1), \ t \in [-r, 0], \\ K_1(t)(\psi(0) - g(x_1)) + \int_0^t (t-s)^{q-1} K_2(t-s) f_1(s) ds, \ t \in J. \end{cases}$$

Now let $U : J \to 2^E$ be a multivalued function defined by

$$U(t) = \{z \in F(t, \tau(t)x_2) : d(z, f_1(t)) = d(f_1(t), F(t, \tau(t)x_2)\}.$$

Since the values of F are compact, then the values of U are nonempty. By applying Lemma (2.2.3), there is a measurable function $f_2 : J \to E$ such that

$$\|f_1(t) - f_2(t)\| = d(f_1(t), F(t, \tau(t)x_2)) \text{ and } f_2(t) \in F(t, \tau(t)x_2), a.e.$$

Note that by [HF$_5$] we get

$$\begin{aligned} \|f_1(t) - f_2(t)\| &= d(f_1(t), F(t, \tau(t)x_2)) \\ &\leq H(F(t, \tau(t)x_1), F(t, \tau(t)x_2)) \\ &\leq k(t) \left\| \tau(t)x_1 - \tau(t)x_1 \right\|_{C_r}, \ a.e. \ t \in J. \end{aligned}$$

Let us define

$$y_2(t) = \begin{cases} \psi(t) - g(x_2), \ t \in [-r, 0], \\ K_1(t)(\psi(0) - g(x_2)) + \int_0^t (t-s)^{q-1} K_2(t-s) f_2(s) ds, \ t \in J. \end{cases}$$

If $t \in [-r, 0]$, then by $[\mathrm{H}_g]$

$$||y_1(t) - y_2(t)|| \leq ||g(x_1) - g(x_2)|| \leq \varsigma ||x_1 - x_2||_{C_b}.$$

If $t \in J$, then

$$\begin{aligned} ||y_1(t) - y_2(t)|| &\leq M||g(x_1) - g(x_2)|| + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} ||f_1(s) - f_2(s)|| ds \\ &\leq M \varsigma ||x_1 - x_2||_{C_b} + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} k(s) \, \|\tau(s)x_1 - \tau(s)x_2\|_{C_r} \, ds \\ &\leq M \varsigma ||x_1 - x_2||_{C_b} + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} k(s) ||x_1 - x_2||_{C_b} ds \\ &\leq ||x_1 - x_2||_{C_b} M \left(\varsigma + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} k(s) ds \right) \\ &\leq ||x_1 - x_2||_{C_b} M \left(\varsigma + \frac{||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}}{\Gamma(q)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}} \right). \end{aligned}$$

By analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H(N_\psi(x_1), N_\psi(x_2)) \leq M \; (\varsigma + \frac{||k||_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}}{\Gamma(q)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}) \, \|x_1 - x_2\|_{C_b}.$$

Invoking to (3.3.1)

$$H(N_\psi(x_1), N_\psi(x_2)) < \rho \, \|x_1 - x_2\|_{C_b}, 0 < \rho < 1$$

This proves that N_Ψ is contraction, and thus, by Lemma (2.2.5), N_ψ has a fixed point which is a mild solution of problem (R_ψ). ■

In the following theorem, we show that there is a continuous selection for the multivalued function $\psi \to S_\psi$.

Theorem 3.3.2 *Let $E = \mathbb{R}$, $F : J \times C_r \to P_{ck}(\mathbb{R})$ and A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t) : t \geq 0\}$ in E. We assume that $[HF_3]$, $[HF_5]$ and the following condition are satisfied:*

[H$_3$] $M\frac{\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}}{\Gamma(q)}\frac{b^{\ q-\sigma}}{\eta^{1-\sigma}}<1,$

where M is a positive number such that $\|T(t)\|_{\mathcal{L}(E,E)} \leq M$ and $\eta = \frac{q-\sigma}{1-\sigma}$.

Then, the multivalued function $\psi \to S_\psi$ has a continuous selection, that is there is a continuous function $u : C_r \to C_b$ such that $u(\psi) \in S_\psi, \forall\psi \in C_r$.

Proof.

According to Theorem (2.3.2), for any $\psi \in C_r$ the set S_ψ is not empty. We construct two sequences of continuous functions $(u_n(\psi))$, $(f_n(\psi))$, $n = 0, 1, 2, ...$ such that:

(i)

$$u_n(\psi) : C_r = C([-r,0],\mathbb{R}) \to C_b = C([-r,b],\mathbb{R}), \quad f_n(\psi) : C_r \to L^1(J,\mathbb{R}),$$

(ii)

$$u_0(\psi)(t) = \begin{cases} \psi(t), & t \in [-r,0], \\ K_1(t)\psi(0), & t \in [0,b], \end{cases}$$

(iii)

$$f_n(\psi)(t) \in F(t, \tau(t)u_n(\psi)), \quad a.e.$$

(iv)

$$u_{n+1}(\psi)(t) = \begin{cases} \psi(t), & t \in [-r,0], \\ K_1(t)\psi(0) + \int_0^t (t-s)^{q-1}K_2(t-s)f_n(\psi)(s)ds, & t \in J, \end{cases}$$

(v)

$$|f_n(\psi)(t) - f_{n-1}(\psi)(t)| \leq k(t)\beta_n(\psi), \quad a.e., \ n \geq 1,$$

where $\beta_0(\psi) = 1 + (M+1)\|\psi\|$ and

$$\beta_{n+1}(\psi) = \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_n(\psi), \ n = 1, 2, ...$$

(vi)

$$d\left(f_n\left(\psi\right)\left(t\right), F\left(t, \tau\left(t\right) u_{n+1}\left(\psi\right)\right)\right) \leq k\left(t\right) \beta_{n+1}\left(\psi\right).$$

In order to define $f_0\left(\psi\right)$, let us define the multifunction:

$$\begin{aligned} F_0^* &: J \times C_r \to P_{ck}\left(\mathbb{R}\right), \\ F_0^*\left(t, \psi\right) &= F\left(t, \tau\left(t\right) u_0\left(\psi\right)\right), \end{aligned}$$

where $u_0\left(\psi\right)$ is given above.

Let us show that, for any fixed $t \in J$, the multivalued function $\psi \to F_0^*\left(t, \psi\right)$ is continuous. Indeed, let $t \in J$ be a fixed point. By [HF$_5$] we have:

$$\begin{aligned} H\left(F_0^*\left(t, \psi_1\right), F_0^*\left(t, \psi_2\right)\right) &= H\left(F\left(t, \tau\left(t\right) u_0\left(\psi_1\right)\right), F\left(t, \tau\left(t\right) u_0\left(\psi_2\right)\right)\right) \\ &\leq k\left(t\right) \left\|\tau\left(t\right) u_0\left(\psi_1\right) - \tau\left(t\right) u_0\left(\psi_2\right)\right\|_{C_r} \\ &= k\left(t\right) \sup_{-r \leq \theta \leq 0} \left|\tau\left(t\right) u_0\left(\psi_1\right)\left(\theta\right) - \tau\left(t\right) u_0\left(\psi_2\right)\left(\theta\right)\right| \\ &\leq k\left(t\right) \sup_{-r \leq \theta \leq 0} \left|\left(u_0\left(\psi_1\right)\right)\left(t+\theta\right) - \left(u_0\left(\psi_2\right)\right)\left(t+\theta\right)\right| \\ &\leq k\left(t\right) \sup_{-r \leq \zeta \leq b} \left|\left(u_0\left(\psi_1\right)\right)\left(\zeta\right) - \left(u_0\left(\psi_2\right)\right)\left(\zeta\right)\right| \end{aligned}$$

then

$$\begin{aligned} & H\left(F_0^*\left(t, \psi_1\right), F_0^*\left(t, \psi_2\right)\right) \\ \leq\ & k(t)\Big[\sup_{-r \leq \zeta \leq 0} \left|\left(u_0\left(\psi_1\right)\right)\left(\zeta\right) - \left(u_0\left(\psi_2\right)\right)\left(\zeta\right)\right| \\ & + \sup_{0 \leq \zeta \leq b} \left|\left(u_0\left(\psi_1\right)\right)\left(\zeta\right) - \left(u_0\left(\psi_2\right)\right)\left(\zeta\right)\right|\Big] \\ \leq\ & k(t)\left[\left\|\psi_1 - \psi_2\right\| + \left\|K_1\left(t\right)\right\| \left|\psi_1\left(0\right) - \psi_2(0)\right|\right] \\ \leq\ & k(t)\left[\left\|\psi_1 - \psi_2\right\| + M\ \left|\psi_1\left(0\right) - \psi_2(0)\right|\right]. \end{aligned}$$

So, $H\left(F_0^*\left(t, \psi_1\right), F_0^*\left(t, \psi_2\right)\right)$ tends to zero, when $\psi_1 \to \psi_2$ in C_r. Hence, for any $t \in J$, the multivalued function $\psi \to F_0^*\left(t, \psi\right)$ is continuous and therefore $l.s.c.$

Now, we define two multivalued functions:

$$G_0 : C_r \to 2^{L^1(J,E)} \ \text{ and } \ H_0 : C_r \to 2^{L^1(J,E)},$$

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where

$$\begin{aligned} G_0(\psi) &= \{v \in L^1(J,E) : v(t) \in F_0^*(t,\psi),\ a.e.\}, \\ H_0(\psi) &= cl\{v \in G_0(\psi) : \|v(t)\| \le k(t)\beta_0(\psi),\ a.e.\}. \end{aligned}$$

Our aim is to prove, by using Lemma (2.2.2), that G_0 is $l.s.c.$ At first, we show that the values of G_0 are decomposable, let $\psi \in C_r$, $v_1, v_2 \in G_0(\psi)$ and A be a Lebesgue measurable subset of J. Then for any $t \in J$

$$(v_1\chi_A + v_2\chi_A)(t) = \begin{cases} v_1(t), & \text{if} \quad t \in A, \\ v_2(t), & \text{if} \quad t \in J \setminus A, \end{cases}$$

Then $(v_1\chi_A + v_2\chi_A) \in G_0(\psi)$ $a.e.$ Hence, the values of G_0 are decomposable.

Note that, from [HF$_5$] for any $\psi \in C_r$ we have:

$$\begin{aligned} d(0, F_0^*(t,\psi)) &\le H(\{0\}, F(t,\tau(t)u_0(\psi)) \\ &\le H(\{0\}, F(t,0)) + H(F(t,0), F(t,\tau(t)u_0(\psi))) \\ &\le k(t) + k(t)\|\tau(t)u_0(\psi)\|_{C_r} \\ &\le k(t)[1 + \sup_{-r\le\theta\le 0} |\tau(t)u_0(\psi)(\theta)|] \\ &\le k(t)[1 + \sup_{-r\le\theta\le 0} |(u_0(\psi))(t+\theta)|] \\ &\le k(t)[1 + \sup_{-r\le\zeta\le b} |(u_0(\psi))(\zeta)|] \end{aligned}$$

then for $a.e.$ $t \in J$

$$\begin{aligned} & d(0, F_0^*(t,\psi)) \\ \le\ & k(t)[1 + \sup_{-r\le\zeta\le 0} \|(u_0(\psi))(\zeta)\| + \sup_{0\le\zeta\le b} \|(u_0(\psi))(\zeta)\|] \\ \le\ & k(t)[1 + \|\psi\| + \|K_1(t)\|\,|\psi(0)|] \\ \le\ & k(t)[1 + \|\psi\| + M\,|\psi(0)|] \\ \le\ & k(t)[1 + (1+M)\|\psi\|] \\ \le\ & k(t)\beta_0(\psi). \end{aligned} \tag{3.3.2}$$

In order to apply Lemma (2.2.2), we define $p_0 : C_r \to L^1(J, \mathbb{R}^{\geq 0})$ by $p_0(\psi)(t) = k(t)\beta_0(\psi)$. We show that p_0 is continuous. Let $\psi_1, \psi_2 \in C_r$. One obtains

$$\begin{aligned}
\|p_0(\psi_1) - p_0(\psi_2)\|_{L^1(J,E)} &= \int_0^b \|p_0(\psi_1)(t) - p_0(\psi_2)(t)\| \, dt \\
&= \int_0^b |k(t)| \left[|\beta_0(\psi_1) - \beta_0(\psi_2)| \right] dt \\
&= \int_0^b |k(t)| \left| 1 + (1+M)\|\psi_1\| - (1 + (1+M)\|\psi_2\|) \right| dt \\
&= \int_0^b |k(t)| \left[(1+M)\|\psi_1 - \psi_2\| \right] dt,
\end{aligned}$$

which implies the continuity of p_0.

Since $C_r = C([-r, 0], \mathbb{R})$ is separable, then from Lemma (2.2.2) and equation (3.3.2) G_0 is $l.s.c.$ Moreover, thanks to [HF$_3$] and [HF$_5$], for any $\psi \in C_r$ the set $S^1_{F(t,\tau(t)\psi)}$ is nonempty and closed, and hence the values of G_0 is nonempty and closed.

Our aim is to show that the multivalued function H_0 satisfies the assumptions of Lemma (2.2.4). So, we show that $H_0(\psi)$ is not empty for any $\psi \in C_r$. So, let $\psi \in C_r$ be a fixed element. Consider the multivalued function Γ^0_ψ defined by:

$$\Gamma^0_\psi(t) = \{x \in F^*_0(t, \psi) : |x - 0| = d(0, F^*_0(t, \psi))\}.$$

By Lemma (2.2.3) Γ^0_ψ is measurable with nonempty values. Then, Γ^0_ψ has a measurable selection, i.e. there is a measurable function $v : J \to \mathbb{R}$ such that $v(t) \in \Gamma^0_\psi(t, \psi)$, for any $t \in J$ and

$$|v(t)| = d(0, F^*_0(t, \psi)),$$

and consequently, by (3.3.2),

$$|v(t)| \leq k(t)\beta_0(\psi), \ a.e.$$

This shows that $v \in H_0(\psi)$.

Therefore, from Lemma (2.2.4), H_0 has a continuous selection $f_0 : C_r \to L^1(J, R)$ such that:

$$f_0(\psi) \in H_0(\psi), \forall \psi \in C_r.$$

Let us show that, for any $\psi \in C_r$, the set

$$L_0(\psi) = \{v \in G_0(\psi) : |v(t)| \leq k(t)\beta_0(\psi),\ a.e.\},$$

is closed. Let (v_n) be a sequence in $L_0(\psi)$ and $v_n \to v$ in $L^1(J, E)$. Since $G_0(\psi)$ is closed, then $v \in G_0(\psi)$. Moreover, $v_n \to v$ in measure, hence there a subsequence (v_{n_k}) of (v_n) such that $v_{n_k} \to v$ almost everywhere. Note that

$$|v_{n_k}(t)| \leq k(t)\beta_0(\psi),\ a.e.$$

This implies to $|v(t)| \leq k(t)\beta_0(\psi)$, $a.e.$ Therefore, $v \in L_0(\psi)$. This shows that $L_0(\psi)$ is closed. Then,

$$f_0(\psi)(t) \in F_0^*(t, \psi) = F(t, \tau(t) u_0(\psi)). \tag{3.3.3}$$

and

$$|f_0(\psi)(t)| \leq k(t)\beta_0(\psi), a.e. \tag{3.3.4}$$

Now, let us define $u_1(\psi) : C_r \to C_b$ as:

$$u_1(\psi)(t) = \begin{cases} \psi(t), & t \in [-r, 0], \\ K_1(t)\psi(0) + \int_0^t (t-s)^{q-1} K_2(t-s) f_0(\psi)(s) ds, & t \in J. \end{cases}$$

In order to define $f_1 : C_r \to L^1(J, \mathbb{R})$, let

$$\begin{aligned} F_1^* &: J \times C_r \to P_{ck}(\mathbb{R}), \\ F_1^*(t, \psi) &= F(t, \tau(t) u_1(\psi)). \end{aligned}$$

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Observe that, for any $t \in J$:

$$\begin{aligned}
H\left(F_1^*\left(t,\psi_1\right),F_1^*\left(t,\psi_2\right)\right) &= H\left(F\left(t,\tau\left(t\right)u_1\left(\psi_1\right)\right),F\left(t,\tau\left(t\right)u_1\left(\psi_2\right)\right)\right)\\
&\leq k\left(t\right)\left\|\tau\left(t\right)u_1\left(\psi_1\right)-\tau\left(t\right)u_1\left(\psi_2\right)\right\|_{C_r}\\
&\leq k\left(t\right)\sup_{-r\leq\theta\leq 0}\left|\tau\left(t\right)u_1\left(\psi_1\right)\left(\theta\right)-\tau\left(t\right)u_1\left(\psi_2\right)\left(\theta\right)\right|\\
&\leq k\left(t\right)\sup_{-r\leq\theta\leq 0}\left|\left(u_1\left(\psi_1\right)\right)\left(t+\theta\right)-\left(u_1\left(\psi_2\right)\right)\left(t+\theta\right)\right|\\
&\leq k(t)\sup_{-r\leq\zeta\leq b}\left|\left(u_1\left(\psi_1\right)\right)\left(\zeta\right)-\left(u_1\left(\psi_2\right)\right)\left(\zeta\right)\right|\\
&\leq k(t)[\sup_{-r\leq\zeta\leq 0}\left|\left(u_1\left(\psi_1\right)\right)\left(\zeta\right)-\left(u_1\left(\psi_2\right)\right)\left(\zeta\right)\right|\\
&\quad+\sup_{0\leq\zeta\leq b}\left|\left(u_1\left(\psi_1\right)\right)\left(\zeta\right)-\left(u_1\left(\psi_2\right)\right)\left(\zeta\right)\right|]\\
&\leq k(t)[\|\psi_1-\psi_2\|+\left|K_1\left(t\right)\right|\left|\psi_1\left(0\right)-\psi_2(0)\right|\\
&\quad+\sup_{0\leq\zeta\leq b}\int_0^\zeta(\zeta-s)^{q-1}\left\|K_2(\zeta-s)\right\|\left|f_0(\psi_1)(s)-f_0(\psi_2)(s)\right|ds]\\
&\leq k(t)[\|\psi_1-\psi_2\|+M|\psi_1\left(0\right)-\psi_2(0)|\\
&\quad+\frac{M}{\Gamma\left(q\right)}\sup_{0\leq\zeta\leq b}\int_0^\zeta(\zeta-s)^{q-1}|f_0(\psi_1)(s)-f_0(\psi_2)(s)|ds]. \quad (3.3.5)
\end{aligned}$$

Note that for any $s \in J$,

$$|f_0(\psi_1)(s)-f_0(\psi_2)(s)| \leq k(t)(\beta_0(\psi_1)+\beta_0(\psi_2)),$$

and this means that $f_0(\psi_1)-f_0(\psi_2) \in L^{\frac{1}{\sigma}}\ (J,\mathbb{R}^+)$. Hence, by the Holder inequality and the continuity of f_0, we conclude that $F_1^*\left(t,\psi\right)$ is continuous and consequently $l.s.c.$ for any $t \in J$.

Now, we define two multivalued functions

$$G_1 : C_r \to 2^{L^1(J,E)} \ \text{ and } \ H_1 : C_r \to 2^{L^1(J,E)}$$

where

$$\begin{aligned}
G_1\left(\psi\right) &= \left\{v \in L^1(J,E) : v(t) \in F_1^*\left(t,\psi\right),\ a.e.\right\},\\
H_1\left(\psi\right) &= cl\left\{v \in G_1\left(\psi\right) : \left|v(t)-f_0\left(\psi\right)\left(t\right)\right| \leq k(t)\beta_1(\psi),\ a.e.\right\},
\end{aligned}$$

where $\beta_1(\psi) = \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_0(\psi)$.

As above we can show that the values of G_1 are closed and decomposable.

Note that from [HF$_5$] for any $\psi \in C_r$ we get:

$$\begin{aligned}
d\left(0, F_1^*(t,\psi)\right) &\leq H\left(\{0\}, F\left(t, \tau(t) u_1(\psi)\right)\right) \\
&\leq H\left(\{0\}, F(t,0)\right) + H\left(F(t,0), F\left(t, \tau(t) u_1(\psi)\right)\right) \\
&\leq k(t) + k(t)\left\|\tau(t) u_1(\psi)\right\|_{C_r} \\
&\leq k(t)[1 + \sup_{-r\leq\theta\leq 0} \left|\tau(t) u_1(\psi)(\theta)\right|] \\
&\leq k(t)[1 + \sup_{-r\leq\theta\leq 0} \left|\left(u_1(\psi)\right)(t+\theta)\right|] \\
&\leq k(t)[1 + \sup_{-r\leq\zeta\leq b} \left|\left(u_1(\psi)\right)(\zeta)\right|] \\
&\leq k(t)[1 + \sup_{-r\leq\zeta\leq 0} \left|\left(u_1(\psi)\right)(\zeta)\right| + \sup_{0\leq\zeta\leq b} \left|\left(u_1(\psi)\right)(\zeta)\right|] \\
&\leq k(t)[1 + \|\psi\| + \|K_1(t)\| \left|\psi(0)\right| \\
&\quad + \sup_{0\leq\zeta\leq b} \int_0^{\zeta} (\zeta - s)^{q-1} \left\|K_2(\zeta - s)\right\| f_0(\psi)(s) \left|ds\right] \\
&\leq k(t)[1 + \|\psi\| + M\left|\psi(0)\right| \\
&\quad + \frac{M}{\Gamma(q)} \sup_{0\leq\zeta\leq b} \int_0^{\zeta} (\zeta - s)^{q-1} \left|f_0(\psi)(s)\right| ds] \\
&\leq k(t)[1 + \|\psi\| + M\left|\psi(0)\right| \\
&\quad + \frac{M}{\Gamma(q)} \sup_{0\leq\zeta\leq b} \int_0^{\zeta} (\zeta - s)^{q-1} k(s)\beta_0(\psi) ds] \\
&\leq k(t)[1 + \|\psi\| + M\ \left|\psi(0)\right| \\
&\quad + \frac{M}{\Gamma(q)} \beta_0(\psi) \sup_{0\leq\zeta\leq b} \int_0^{\zeta} (\zeta - s)^{q-1} k(s) ds]. \qquad (3.3.6)
\end{aligned}$$

By Holder inequality, we have:

$$\begin{aligned}
\int_0^{\zeta} (\zeta - s)^{q-1} k(s) ds &\leq \left(\int_0^{\zeta} (\zeta - s)^{\frac{q-1}{1-\sigma}} ds\right)^{1-\sigma} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\
&\leq \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.
\end{aligned}$$

Thus, the equation (3.3.6) becomes:

$$d\left(0, F_1^*(t,\psi)\right) \leq k(t)[1 + \|\psi\| + M\left|\psi(0)\right| + \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_0(\psi)].$$

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Let $p_1 : C_r \to L^1(J, \mathbb{R})$ defined by:

$$p_1(\psi)(t) = k(t)[1 + \|\psi\| + M|\psi(0)| + \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_0(\psi)].$$

By the same method used to prove that p_0 is continuous, we can show that p_1 is continuous. Then from Lemma (2.2.2) G_1 is $l.s.c.$ with nonempty closed values.

Now, let $\psi \in C_r$ be fixed. Let us show that $H_1(\psi)$ is not empty. At first, note that from (3.3.3), for any $t \in J$, we have:

$$\begin{aligned}
d(f_0(\psi)(t), F_1^*(t,\psi)) &\leq H(F_0^*(t,\psi), F_1^*(t,\psi)) \\
&= H(F(t, \tau(t)u_0(\psi)), F(t, \tau(t)u_1(\psi))) \\
&\leq k(t)\|\tau(t)u_0(\psi) - \tau(t)u_1(\psi)\|_{C_r} \\
&\leq k(t) \sup_{-r\leq\theta\leq 0} |\tau(t)u_0(\psi)(\theta) - \tau(t)u_1(\psi)(\theta)| \\
&\leq k(t) \sup_{-r\leq\theta\leq 0} |(u_0(\psi))(t+\theta) - (u_1(\psi))(t+\theta)| \\
&\leq k(t) \sup_{-r\leq\zeta\leq b} |(u_0(\psi))(\zeta) - (u_1(\psi))(\zeta)| \\
&\leq k(t)[\sup_{-r\leq\zeta\leq 0} |(u_0(\psi))(\zeta) - (u_1(\psi))(\zeta)| \\
&\quad + \sup_{0\leq\zeta\leq b} |(u_0(\psi))(\zeta) - (u_1(\psi))(\zeta)|] \\
&\leq k(t)[\sup_{-r\leq\zeta\leq 0} |\psi(\zeta) - \psi(\zeta)| \\
&\quad + \sup_{0\leq\zeta\leq b} \int_0^\zeta (\zeta - s)^{q-1} \|K_2(\zeta - s)\| |f_0(\psi)(s)| ds] \\
&\leq k(t) \frac{M}{\Gamma(q)} \sup_{0\leq\zeta\leq b} \int_0^\zeta (\zeta - s)^{q-1} k(s)\beta_0(\psi) ds \\
&\leq k(t) \frac{M}{\Gamma(q)} \beta_0(\psi) \sup_{0\leq\zeta\leq b} \int_0^\zeta (\zeta - s)^{q-1} k(s) ds,
\end{aligned}$$

By Holder inequality we get:

$$\begin{aligned}
\int_0^\zeta (\zeta - s)^{q-1} k(s) ds &\leq \left(\int_0^\zeta (\zeta - s)^{\frac{q-1}{1-\sigma}} ds\right)^{1-\sigma} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\
&\leq \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.
\end{aligned}$$

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Then

$$d\left(f_0\left(\psi\right)(t), F_1^*\left(t,\psi\right)\right) \leq k(t)\frac{M}{\Gamma\left(q\right)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\left\|k\right\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_0(\psi) \leq k(t)\beta_1(\psi), \quad (3.3.7)$$

where $\beta_1(\psi) = \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\left\|k\right\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_0(\psi)$.

Secondly, let Γ_ψ^1 be a multifunction defined by:

$$\Gamma_\psi^1(t) = \{x \in F_1^*(t,\psi) : |x - f_0(\psi)(t)| = d(f_0(\psi)(t), F_1^*(t,\psi))\}.$$

Then by Lemma (2.2.3), $\Gamma_\psi^1(t)$ is a measurable and hence there is a measurable function $v : J \to E$ such that $v(t) \in \Gamma_\psi^1(t)$, $\forall t \in J$. This implies to

$$|v(t) - f_0(\psi)(t)| = d(f_0(\psi)(t), F_1^*(t,\psi)).$$

This equation with (3.3.7) give us:

$$|v(t) - f_0(\psi)(t)| \leq k(t)\beta_1(\psi), a.e.$$

and $v \in G_1(\psi)$. Thus $v \in H_1(\psi)$.

From Lemma (2.2.4), H_1 has a continuous selection $f_1 : C_r \to L^1(J,\mathbb{R})$ such that:

$$f_1(\psi) \in H_1(\psi), \forall \psi \in C_r.$$

By arguing as above, we can show that the closedness of $G_1(\psi)$ implies that the set

$$L_1(\psi) = \{v \in G_1(\psi) : |v(t) - f_0(\psi)(t)| \leq k(t)\beta_1(\psi), \quad a.e.\}.$$

is closed. Hence,

$$f_1(\psi)(t) \in F_1^*(t,\psi) = F(t,\tau(t)u_1(\psi)),$$

and

$$|f_1(\psi)(t) - f_0(\psi)(t)| \leq k(t)\beta_1(\psi), a.e. \quad (3.3.8)$$

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Suppose that we have constructed $u_0, u_1, ..., u_n, ..., f_0, f_1, ..., f_n, ...$ satisfying (i)→(vi).

Let us define $u_{n+1} : C_r \to C_b$ as:

$$u_{n+1}(\psi)(t) = \begin{cases} \psi(t), & t \in [-r, 0], \\ K_1(t)\psi(0) + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(\psi)(s) ds, & t \in J, \end{cases}$$

and hence define:

$$F^*_{n+1} : J \times C_r \to P_{ck}(\mathbb{R}), \ by :$$

$$F^*_{n+1}(t, \psi) = F(t, \tau(t) u_{n+1}(\psi)),$$

$$G_{n+1} : C_r \to 2^{L^1(J,\mathbb{R}}, \ by :$$

$$G_{n+1}(\psi) = \left\{ v \in L^1(J, E) : v(t) \in F^*_{n+1}(t, \psi), \quad a.e. \right\},$$

and

$$H_{n+1} : C_r \to 2^{L^1(J,\mathbb{R})}, \ by :$$

$$H_{n+1}(\psi) = cl \left\{ v \in G_{n+1}(\psi) : |v(t) - f_n(\psi)(t)| \le k(t)\beta_n(\psi), \quad a.e. \right\},$$

where $\beta_{n+1}(\psi) = \frac{M}{\Gamma(q)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \beta_n(\psi), n \ge 0.$

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Let us show that for any $t \in J$, $t \to F^*_{n+1}(t,\psi)$ is continuous. So, let $\psi_1, \psi_2 \in C_r$.

By $[\mathrm{HF}_5]$, we have:

$$\begin{aligned}
H\left(F^*_{n+1}(t,\psi_1), F^*_{n+1}(t,\psi_2)\right) &= H\left(F\left(t,\tau(t)\,u_{n+1}(\psi_1)\right), F\left(t,\tau(t)\,u_{n+1}(\psi_2)\right)\right) \\
&\leq k(t)\left\|\tau(t)\,u_{n+1}(\psi_1) - \tau(t)\,u_{n+1}(\psi_2)\right\|_{C_r} \\
&\leq k(t) \sup_{-r\leq\theta\leq 0} \left|\tau(t)\,u_{n+1}(\psi_1)(\theta) - \tau(t)\,u_{n+1}(\psi_2)(\theta)\right| \\
&\leq k(t) \sup_{-r\leq\theta\leq 0} \left|(u_{n+1}(\psi_1))(t+\theta) - (u_{n+1}(\psi_2))(t+\theta)\right| \\
&\leq k(t) \sup_{-r\leq\zeta\leq b} \left|(u_{n+1}(\psi_1))(\zeta) - (u_{n+1}(\psi_2))(\zeta)\right| \\
&\leq k(t)[\sup_{-r\leq\zeta\leq 0} \left|(u_{n+1}(\psi_1))(\zeta) - (u_{n+1}(\psi_2))(\zeta)\right| \\
&\quad + \sup_{0\leq\zeta\leq b} \left|(u_{n+1}(\psi_1))(\zeta) - (u_{n+1}(\psi_2))(\zeta)\right|] \\
&\leq k(t)[\|\psi_1 - \psi_2\| + \|K_1(t)\|\,|\psi_1(0) - \psi_2(0)| \\
&\quad + \sup_{0\leq\zeta\leq b}\int_0^\zeta (\zeta-s)^{q-1}\,\|K_2(\zeta-s)\|\,|f_n(\psi_1)(s) - f_n(\psi_2)(s)|ds \\
&\leq k(t)[\|\psi_1 - \psi_2\| + M|\psi_1(0) - \psi_2(0)| \\
&\quad + \frac{M}{\Gamma(q)}\sup_{0\leq\zeta\leq b}\int_0^\zeta (\zeta-s)^{q-1}|f_n(\psi_1)(s) - f_n(\psi_2)(s)|ds].
\end{aligned}$$

Hence by the continuity of f_n, we infer that the multivalued function $t \to F^*_{n+1}(t,\psi)$ is continuous and therefore $l.s.c.$

Moreover, from $[\mathrm{H}_2]$ for any $\psi \in C_r$ we get:

$$\begin{aligned}
d\left(0, F^*_{n+1}(t,\psi)\right) &\leq H\left(\{0\}, F\left(t,\tau(t)\,u_{n+1}(\psi)\right)\right) \\
&\leq H\left(\{0\}, F(t,0)\right) + H\left(F(t,0), F\left(t,\tau(t)\,u_{n+1}(\psi)\right)\right) \\
&\leq k(t) + k(t)\left\|\tau(t)\,u_{n+1}(\psi)\right\|_{C_r} \\
&\leq k(t)[1 + \sup_{-r\leq\theta\leq 0}\left|\tau(t)\,u_{n+1}(\psi)(\theta)\right|] \\
&\leq k(t)[1 + \sup_{-r\leq\theta\leq 0}\left|(u_{n+1}(\psi))(t+\theta)\right|] \\
&\leq k(t)[1 + \sup_{-r\leq\zeta\leq b}\left|(u_{n+1}(\psi))(\zeta)\right|]
\end{aligned}$$

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then

$$\begin{aligned}
& d\left(0, F_{n+1}^{*}(t,\psi)\right) \\
\leq\ & k(t)[1+\sup_{-r\leq\zeta\leq 0}|(u_{n+1}(\psi))(\zeta)|+\sup_{0\leq\zeta\leq b}|(u_{n+1}(\psi))(\zeta)|] \\
\leq\ & k(t)[1+\|\psi\|+\|K_1(t)\|\,|\psi(0)| \\
& +\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}\|K_2(\zeta-s)\|\,|f_n(\psi)(s)|ds] \\
\leq\ & k(t)[1+\|\psi\|+M|\psi(0)| \\
& +\frac{M}{\Gamma(q)}\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}|f_n(\psi)(s)|ds]. \qquad (3.3.9)
\end{aligned}$$

Further more, from property (v), for any $t\in J$, one obtains:

$$\begin{aligned}
|f_n(\psi)(t)| \leq\ & |f_n(\psi)(t)-f_{n-1}(\psi)(t)|+|f_{n-1}(\psi)(t)-f_{n-2}(\psi)(t)|+.... \\
& +|f_1(\psi)(t)-f_0(\psi)(t)|+|f_0(\psi)(t)| \\
\leq\ & k(t)\beta_n(\psi)+k(t)\beta_{n-1}(\psi)+......+k(t)\beta_1(\psi)+k(t)\beta_0(\psi).
\end{aligned}$$

Thus, the equation (3.3.9) becomes:

$$d\left(0, F_{n+1}^{*}(t,\psi)\right)\leq k(t)[1+\|\psi\|+M|\psi(0)|+\frac{M}{\Gamma(q)}\sum_{m=0}^{n}\beta_m(\psi)\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}k(s)ds]. \qquad (3.3.10)$$

Note that, by Holder inequality, we have:

$$\begin{aligned}
\int_0^{\zeta}(\zeta-s)^{q-1}k(s)ds \leq\ & \left(\int_0^{\zeta}(\zeta-s)^{\frac{q-1}{1-\sigma}}ds\right)^{1-\sigma}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\
\leq\ & \frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.
\end{aligned}$$

Then, by (3.3.10):

$$\begin{aligned}
d\left(0, F_{n+1}^{*}(t,\psi)\right) \leq\ & k(t)[1+\|\psi\|+M|\psi(0)|+\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\left(\sum_{m=0}^{n}\beta_m(\psi)\right)] \\
\leq\ & p_{n+1}(\psi)(t), \qquad (3.3.11)
\end{aligned}$$

where $p_{n+1}:C_r\to L^1(J,\mathbb{R}^{\geq 0})$ and defined by:

$$p_{n+1}(\psi)(t)=k(t)[1+\|\psi\|+M|\psi(0)|+\frac{M}{\Gamma(q)}\left(\sum_{m=0}^{n}\beta_m(\psi)\right)\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}].$$

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Observe that for $\psi_1, \psi_2 \in C_r$ we have:

$$
\begin{aligned}
\left\|p_{n+1}(\psi_1) - p_{n+1}(\psi_2)\right\|_{L^1(J,\mathbb{R})} &= \int_0^b \left|p_{n+1}(\psi_1)(t) - p_{n+1}(\psi_2)(t)\right| dt \\
&= \int_0^b k(t)\{\|\psi_1 - \psi_2\| + M\ |\psi_1(0) - \psi_2(0)| \\
&\quad + \frac{M}{\Gamma(q)}\left|\left(\sum_{m=0}^{n}\beta_m(\psi_1)\right) - \left(\sum_{m=0}^{n}\beta_m(\psi_2)\right)\right| \frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\}dt,
\end{aligned}
$$

This shows that p_{n+1} is continuous.

Then from Lemma (2.2.2) and (3.3.11) G_{n+1} is $l.s.c.$ with nonempty closed values. Moreover, we can show as above that the values of G_{n+1} are decomposable.

In order to apply Lemma (2.2.4) we prove that the values of H_{n+1} are not empty. So, let $\psi \in C_r$ be fixed.

For any $t \in J$, by [HF$_5$], one obtains:

$$
\begin{aligned}
d\left(f_n(\psi)(t), F^*_{n+1}(t,\psi)\right) &\leq H\left(F^*_n(t,\psi), F^*_{n+1}(t,\psi)\right) \\
&= H\left(F\left(t, \tau(t)u_n(\psi)\right), F\left(t, \tau(t)u_{n+1}(\psi)\right)\right) \\
&\leq k(t)\left\|\tau(t)u_n(\psi) - \tau(t)u_{n+1}(\psi)\right\|_{C_r} \\
&\leq k(t)\sup_{-r\leq\theta\leq 0}\left\|\tau(t)u_n(\psi)(\theta) - \tau(t)u_{n+1}(\psi)(\theta)\right\| \\
&\leq k(t)\sup_{-r\leq\theta\leq 0}\left\|(u_n(\psi))(t+\theta) - (u_{n+1}(\psi))(t+\theta)\right\| \\
&\leq k(t)\sup_{-r\leq\zeta\leq b}\left\|(u_n(\psi))(\zeta) - (u_{n+1}(\psi))(\zeta)\right\|.
\end{aligned}
$$

Then,

$$\begin{aligned}
& d\left(f_n(\psi)(t), F_{n+1}^*(t,\psi)\right) \\
\leq\ & k(t)[\sup_{-r\leq\zeta\leq 0}\|(u_n(\psi))(\zeta)-(u_{n+1}(\psi))(\zeta)\| \\
& +\sup_{0\leq\zeta\leq b}\|(u_n(\psi))(\zeta)-(u_{n+1}(\psi))(\zeta)\|] \\
\leq\ & k(t)[\|\psi(t)-\psi(t)\|+\|K_1(t)\|\,\|\psi(0)-\psi(0)\| \\
& +\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}\|K_2(\zeta-s)\|\,|f_n(\psi)(s)-f_{n-1}(\psi)(s)|ds] \\
\leq\ & k(t)\frac{M}{\Gamma(q)}\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}|f_n(\psi)(s)-f_{n-1}(\psi)(s)|ds \\
\leq\ & k(t)\frac{M}{\Gamma(q)}\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}k(s)\beta_n(\psi)ds \\
\leq\ & k(t)\frac{M}{\Gamma(q)}\beta_n(\psi)\sup_{0\leq\zeta\leq b}\int_0^{\zeta}(\zeta-s)^{q-1}k(s)ds,
\end{aligned}$$

By Holder inequality we get:

$$\begin{aligned}
\int_0^{\zeta}(\zeta-s)^{q-1}k(s)ds \ &\leq\ \left(\int_0^{\zeta}(\zeta-s)^{\frac{q-1}{1-\sigma}}\right)^{1-\sigma}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\
&\leq\ \frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.
\end{aligned}$$

Then

$$\begin{aligned}
d\left(f_n(\psi)(t), F_{n+1}^*(t)\right) \ &\leq\ k(t)\frac{M}{\Gamma(q)}\beta_n(\psi)\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)} \\
&\leq\ k(t)\beta_{n+1}(\psi), \qquad (3.3.12)
\end{aligned}$$

where $\beta_{n+1}(\psi)=\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_n(\psi)$.

Now, let Γ_{ψ}^{n+1} be a multivalued function defined on J by:

$$\Gamma_{\psi}^{n+1}(t)=\{x\in F_{n+1}^*(t,\psi):|x-f_n(\psi)(t)|=d(f_n(\psi)(t),F_{n+1}^*(t,\psi))\}.$$

Since the values of F_{n+1}^* are compact, then by Lemma (2.2.3) there is a measurable function $v:J\to E$ such that $v(t)\in\Gamma_{\psi}^{n+1}(t),\forall t\in J$.

Therefore, by (3.3.12)

$$\begin{aligned} |v(t) - f_n(\psi)(t)| &= d(f_n(\psi), F^*_{n+1}(t,\psi)) \\ &\leq k(t)\beta_{n+1}(\psi), \ a.e. \end{aligned}$$

Then $v \in G_{n+1}(\psi)$, $|v(t) - f_n(\psi)(t)| \leq k(t)\beta_{n+1}(\psi)$, $a.e.$ and consequently $v \in H_{n+1}(\psi)$.

From Lemma (2.2.4), H_{n+1} has a continuous selection $f_{n+1} : C_r \to L^1(J,\mathbb{R})$ such that $f_{n+1}(\psi) \in H_{n+1}(\psi), \forall \psi \in C_r$.

By arguing as above, we can show that the closedness of $G_{n+1}(\psi)$ implies that the set

$$L_{n+1}(\psi) = \{v \in G_{n+1}(\psi), |v(t) - f_n(\psi)(t)| \leq k(t)\beta_n(\psi)\}$$

is closed. Then

$$f_{n+1}(\psi)(t) \in F^*_{n+1}(t,\psi) = F(t, \tau(t) u_{n+1}(\psi)),$$

and

$$|f_{n+1}(\psi)(t) - f_n(\psi)(t)| \leq k(t)\beta_{n+1}(\psi). \tag{3.3.13}$$

Therefore, the functions $u_0, u_1, ...u_n, ..., f_0, f_1, ..., f_n$, ...are constructed and satisfying the properties (i)→(vi).

Now, from the property (v), for all $t \in J$ and for all $\psi \in C_r$ we have:

$$\begin{aligned}
|f_{n+1}(\psi)(t) - f_n(\psi)(t)| &\leq k(t)\beta_{n+1}(\psi) \\
&= k(t)\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\beta_n(\psi) \\
&= k(t)\left[\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\right]^2\beta_{n-1}(\psi) \\
&. \\
&. \\
&. \\
&= k(t)\left[\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\right]^{n+1}\beta_0(\psi) \\
&= k(t)\left[\frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}\right]^{n+1}(1+(M+1)\|\psi\|).
\end{aligned} \tag{3.3.14}$$

Then

$$\|f_{n+1}(\psi)(t) - f_n(\psi)(t)\|_{L^1(J,E)} \leq k(t)(1+(M+1)\|\psi\|)\delta^{n+1}, \tag{3.3.15}$$

where

$$\delta = \frac{M}{\Gamma(q)}\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}^+)}.$$

This implies that for any $\psi \in C_r$, any two natural number n, m with $n < m$ and any $t \in J$

$$\begin{aligned}
||f_m(\psi) - f_n(\psi)||_{L^1(J,\mathbb{R})} &\leq ||f_m(\psi) - f_{m-1}(\psi)||_{L^1(J,\mathbb{R})} + \ldots.. + \|f_{n+1}(\psi) - f_n(\psi)\|_{L^1(J,\mathbb{R})} \\
&\leq ||k||_{L^1(J,\mathbb{R}^+)}[\delta^m + \delta^{m-1} + \ldots.. + \delta^{n+1}](1+(M+1)\|\psi\|) \\
&\leq ||k||_{L^1(J,\mathbb{R}^+)}\delta^{n+1}[1 + \delta + \delta^2 + \ldots.. + \delta^{m-(n+1)}](1+(M+1)\|\psi\|) \\
&\leq ||k||_{L^1(J,\mathbb{R}^+)}\delta^{n+1}\sum_{k=0}^{\infty}\delta^k(1+(M+1)\|\psi\|) \\
&= ||k||_{L^1(J,\mathbb{R}^+)}\delta^{n+1}\frac{1}{1-\delta}(1+(M+1)\|\psi\|).
\end{aligned} \tag{3.3.16}$$

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Since $0 < \delta < 1$, then:

$$\lim_{m,n\to\infty} \|f_m(\psi) - f_n(\psi)\|_{L^1(J,\mathbb{R})} = 0,$$

This implies that for any $\psi \in C_r$, the sequence $(f_n(\psi))$ is Cauchy in $L^1(J,E)$. So, there exists a function $f : C_r \to L^1(J,\mathbb{R})$ such that:

$$\lim_{n\to\infty} f_n(\psi) = f(\psi), \quad \forall\psi \in C_r.$$

To prove that $f : C_r \to L^1(J,\mathbb{R})$ is continuous, let $\psi_1, \psi_2 \in C_r$ and $\varepsilon > 0$, since $f_n(\psi_1) \to f(\psi_1)$ and $f_n(\psi_2) \to f(\psi_2)$, there is a natural number $N = N(\psi_1, \psi_2)$ such that for $n \geq N$ we have:

$$\|f_n(\psi_1) - f(\psi_1)\|_{L^1(J,\mathbb{R})} \leq \frac{\varepsilon}{3}, \tag{3.3.17}$$

and

$$\|f_n(\psi_2) - f(\psi_2)\|_{L^1(J,\mathbb{R})} \leq \frac{\varepsilon}{3}. \tag{3.3.18}$$

By the continuity of f_N, there is $\delta > 0$ such that:

$$\|\psi_1 - \psi_2\| < \delta \Rightarrow \|f_N(\psi_1) - f_N(\psi_2)\|_{L^1(J,\mathbb{R})} < \frac{\varepsilon}{3}. \tag{3.3.19}$$

Then from (3.3.17), (3.3.18), (3.3.19) we have:

$$\begin{aligned}\|f(\psi_1) - f(\psi_2)\|_{L^1(J,\mathbb{R})} &\leq \|f(\psi_1) - f_N(\psi_1)\|_{L^1(J,\mathbb{R})} + \|f_N(\psi_1) - f_N(\psi_2)\|_{L^1(J,\mathbb{R})} \\ &\quad + \|f_N(\psi_2) - f(\psi_2)\|_{L^1(J,E)} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

This means that $f : C_r \to L^1(J,\mathbb{R})$ is continuous.

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Now, let $\psi \in C_r$ be fixed. From the definition of $u_n(\psi)$ and (3.3.15) we get:

$$\begin{aligned}
\|u_{n+1}(\psi)-u_n(\psi)\|_{C_b} &\leq \sup_{-r\leq t\leq 0}|u_{n+1}(\psi)(t)-u_n(\psi)(t)|+\sup_{0\leq t\leq b}|u_{n+1}(\psi)(t)-u_n(\psi)(t)| \\
&\leq \sup_{0\leq t\leq b}\int_0^t (t-s)^{q-1}\|K_2(t-s)\|\,|f_n(\psi)(s)-f_{n-1}(\psi)(s)|ds \\
&\leq \frac{M}{\Gamma(q)}\sup_{0\leq t\leq b}\int_0^t (t-s)^{q-1}|f_n(\psi)(s)-f_{n-1}(\psi)(s)|ds \\
&\leq \frac{M}{\Gamma(q)}\delta^n(1+(M+1)\|\psi\|)\sup_{0\leq t\leq b}\int_0^t (t-s)^{q-1}k(s)ds \\
&\leq \frac{M}{\Gamma(q)}\delta^n(1+(M+1)\|\psi\|)\frac{b^{q-\sigma}}{\eta^{1-\sigma}}\|k\|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}+)}. \qquad (3.3.20)
\end{aligned}$$

By arguing as in (3.3.16) we can show that $u_n(\psi)$ is a Cauchy sequence in C_b. Hence there is $u(\psi)\in C_b$ such that $u_n(\psi)$ converges to $u(\psi)$ in C_b.

Let us define $u: C_r \to C_b$ such that

$$u(\psi)=\lim_{n\to\infty}u_n(\psi).$$

To prove that $u: C_r \to C_b$ is continuous, we take $\psi_1,\psi_2\in C_r$. Then for any $n\geq 1$ we have:

$$\|u(\psi_1)-u(\psi_2)\|_{C_b}\leq \|u(\psi_1)-u_n(\psi_1)\|_{C_b}+\|u_n(\psi_1)-u_n(\psi_2)\|_{C_b}+\|u_n(\psi_2)-u(\psi_2)\|_{C_b}.$$

It follows from the continuity of $u_n(\psi)$ and from the fact that $u_n(\psi)$ converges to $u(\psi)$ in C_b that u is continuous.

Next, we prove that

$$f(\psi)(t)\in F(t,\tau(t)u(\psi)),\, a.e.$$

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So, $\psi \in C_r$. We have by [H$_2$]:

$$\begin{aligned}
d\left(f_n(\psi)(t), F(t,\tau(t)u(\psi))\right) &\leq H\left(F(t,\tau(t)u_n(\psi)), F(t,\tau(t)u(\psi))\right)\\
&\leq k(t)\left\|\tau(t)u_n(\psi)-\tau(t)u(\psi)\right\|_{C_r}\\
&\leq k(t)\sup_{-r\leq\theta\leq 0}\left|\tau(t)u_n(\psi)(\theta)-\tau(t)u(\psi)(\theta)\right|\\
&\leq k(t)\sup_{-r\leq\theta\leq 0}\left|(u_n(\psi))(t+\theta)-(u(\psi))(t+\theta)\right|\\
&\leq k(t)\sup_{-r\leq\zeta\leq b}\left|(u_n(\psi))(\zeta)-(u(\psi))(\zeta)\right|\\
&\leq k(t)\left\|u_n(\psi)-u(\psi)\right\|_{C_b}, \, a.e.
\end{aligned}$$

Since $f_n(\psi)$ converges to $f(\psi)$ in $L^1(J,\mathbb{R})$, then $f_n(\psi)$ converges in measure to $f(\psi)$ and hence we can find a subsequence $f_{n_k}(\psi)$ of $f_n(\psi)$ such that

$$f_{n_k}(\psi) \to f(\psi), \ a.e.$$

So, the last inequality with (3.3.20) gives us:

$$f(\psi)(t) \in F(t,\tau(t)u(\psi)), \ a.e. \tag{3.3.21}$$

Now, let $v : C_r \to C_b$ defined by:

$$v(\psi)(t) = \begin{cases} \psi(t), & t \in [-r,0], \\ K_1(t)\psi(0) + \int_0^t (t-s)^{q-1}K_2(t-s)f(\psi)(s)ds, & t \in J. \end{cases}$$

Let us show that

$$u(\psi)(t) = v(\psi)(t), \ \forall \psi \in C_r \text{ and } t \in J.$$

Let $\psi \in C_r$. Note that for almost $t \in J$,

$$\begin{aligned}
|f_n(\psi)(t)| &\leq H(\{0\}, F(t,\tau(t)u_n(\psi)))\\
&\leq H(\{0\}, F(t,0)) + H(F(t,0), F(t,\tau(t)u_n(\psi)))\\
&\leq k(t) + k(t)\left\|\tau(t)u_n(\psi)\right\|_{C_r}\\
&\leq k(t) + k(t)\left\|u_n(\psi)\right\|_{C_b}.
\end{aligned}$$

Since $u_n(\psi)$ converges uniformly to $u(\psi)$ in C_b, then $u_n(\psi)$ is uniformly bounded, hence we can find an integrable function $z_\psi : J \to [0, \infty)$ such that

$$\|f_n(\psi)(t)\| \leq z_\psi(t), \ a.e.$$

Moreover, as above, there is a subsequence $(f_{n_n}(\psi))$ of $f_n(\psi)$ such that

$$f_{n_k}(\psi) \to f(\psi), \ a.e.$$

Then, by the Lebesgue dominated convergence theorem we get for $t \in J$,

$$\lim_{n_k \to \infty} u_{n_k}(\psi)(t) = v(\psi)(t).$$

Then

$$v(\psi)(t) = u(\psi)(t), \ \forall t \in J \text{ and } \psi \in C_r.$$

Thus:

$$u(\psi)(t) = \begin{cases} \psi(t), & t \in [-r, 0], \\ K_1(t)\psi(0) + \int_0^t (t-s)^{q-1} K_2(t-s) f(\psi)(s) ds, & t \in J, \end{cases}$$

and

$$u(\psi) \in S(\psi), \forall \psi \in C_r.$$

This means that $u : C_r \to C_b$ is a continuous selection for S_ψ and this complete the proof. ■

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